On axiomatizations of (non-)contingency logic

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1. Introduction

From Aristotle...

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- necessity

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- possibility

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- contingency

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- necessity
- possibility
- contingency

Three kinds of propositions: necessary proposition, impossible proposition, contingent proposition.

Contingency: an example

- Will there be necessarily sea battles tomorrow? No!
- Will there be necessarily no sea battles tomorrow? No!
- Why is it the case?
- The proposition "there will be sea battles tomorrow" (P) is **contingent**, i.e., it is possible that P and it is possible that not P.

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- ...

Axiomatizations: 1966-2015

Frames	Known results
$\mathcal{K}$	[Humberstone, 1995, Kuhn, 1995, van der Hoek and Lomuscio, 2004]
$\mathcal{D}$	[Humberstone, 1995]
$\mathcal{T}$	[Montgomery and Routley, 1966]
4	[Kuhn, 1995]
5	[Zolin, 1999]
$\mathcal{B}$	[Fan et al., 2014, Fan et al., 2015]

Language

$$\mathcal{L}(\Delta) \qquad \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Delta\varphi$$

- $\Delta\varphi$: it is non-contingent that φ.
- $\nabla\varphi =_{df} \neg\Delta\varphi$: it is contingent that φ.

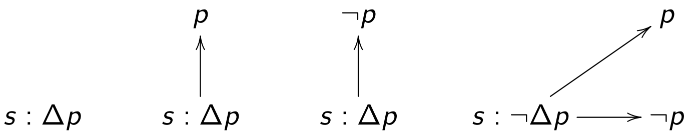
2. Kripke semantics and Axiomatizations

Kripke Semantics

$$\mathcal{M}, s \vDash \Delta\varphi \quad \text{iff} \quad \forall t_1, t_2 \in R(s) : (\mathcal{M}, t_1 \vDash \varphi \text{ iff } \mathcal{M}, t_2 \vDash \varphi)$$

$$\mathcal{M}, s \vDash \Delta\varphi \quad \text{iff} \quad \forall t_1, t_2 \in R(s) : \mathcal{M}, t_1 \vDash \varphi \text{ implies } \mathcal{M}, t_2 \vDash \varphi.$$

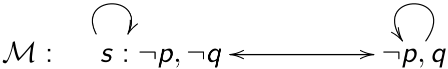
$$\mathcal{M}, s \vDash \Delta\varphi \quad \text{iff} \quad \forall t \in R(s) : \mathcal{M}, t \vDash \varphi, \text{ or, } \forall t \in R(s) : \mathcal{M}, t \nvDash \varphi.$$



Non-normality

Even on **S5**-models,

$$\not\models \Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$$



$\mathcal{M}, s \models \Delta(p \to q) \wedge \Delta p$, but $\mathcal{M}, s \not\models \Delta q$.
Even non-monotonic: $\not\models \Delta p \to \Delta(p \vee q)$.

Expressivity [Fan et al., 2015]

Proposition

$\mathcal{L}(\Delta)$ is less expressive than standard modal logic on $\mathcal{K}$, $\mathcal{D}$, $\mathcal{B}$, 4, 5.

Proposition

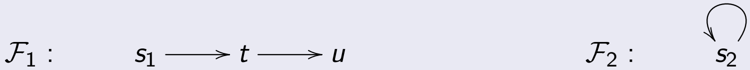
$\mathcal{L}(\Delta)$ is equally expressive as standard modal logic on $\mathcal{T}$.

Frame definability [Zolin, 1999, Fan et al., 2015]

Proposition

The frame properties of seriality, reflexivity, symmetry, transitivity and Euclidicity are not definable in $\mathcal{L}(\Delta)$.

Sketch.



□

A summary

- Non-normal
- Less expressive
- Cannot define many usual frame properties

These make the axiomatizations of $\mathcal{L}(\Delta)$ non-trivial

Montgomery and Routley's reflexive logics

$\mathbf{T}^{\Delta}$

TAUT	all instances of tautologies
ΔEqu	$\Delta\varphi \leftrightarrow \Delta\neg\varphi$
ΔT	$\varphi \to (\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi))$
MP	From φ and $\varphi \to \psi$ infer ψ
NECΔ	From φ infer $\Delta\varphi$
REΔ	From $\varphi \leftrightarrow \psi$ infer $\Delta\varphi \leftrightarrow \Delta\psi$

$\mathbf{S4}^{\Delta} = \mathbf{T}^{\Delta} + (\Delta\varphi \to \Delta\Delta\varphi)$
$\mathbf{S5}^{\Delta} = \mathbf{T}^{\Delta} + \Delta\Delta\varphi$
NB: $\mathcal{L}(\Delta)$ is equally expressive as $\mathcal{L}(\Box)$ on $\mathcal{T}$, since
$\Delta\varphi \overset{def}{=} \Box\varphi \vee \Box\neg\varphi$ and $\Box\varphi \overset{def}{=} \Delta\varphi \wedge \varphi$.
But ΔT cannot be obtained from $\Box\varphi \to \varphi$ via translation!

A question

Q: how to axiomatize $\mathcal{L}(\Delta)$ over other classes of frames?

'Simulate' the canonical relation in standard modal logic.
$xR^c y$ iff $\lambda(x) \subseteq y$, where $\lambda(x) = \{\varphi \mid \Box\varphi \in x\}$.

How?

Humberstone's logic

$$\lambda(x) = \{\varphi \mid \Delta\varphi \in x \text{ and for all } \psi, \vdash \varphi \to \psi \text{ implies } \Delta\psi \in x\}.$$

NC:

$$(\Delta\neg) \qquad \Delta\neg\varphi \to \Delta\varphi$$

$$(\mathrm{NCR})_k \qquad \frac{s_1(\varphi_1, \cdots, \varphi_k) \to \psi_1 \quad \cdots \quad s_{2^k}(\varphi_1, \cdots, \varphi_k) \to \psi_{2^k}}{(\Delta\varphi_1 \wedge \cdots \wedge \Delta\varphi_k) \to (\Delta\psi_1 \vee \cdots \vee \Delta\psi_{2^k})}$$

- $k \in \mathbb{N}$, infinitary system
- Question: finitely axiomatizable?

Kuhn's logics

- $\lambda(x) = \{\varphi \mid \Delta(\varphi \vee \psi) \in x \text{ for all } \psi\}$
- $\mathbf{K\Delta}$ = minimal non-contingency logic

PL	All substitution instances of tautologies
A1	$\Delta\neg\varphi \to \Delta\varphi$
A2	$\Delta\varphi \wedge \nabla(\varphi \wedge \psi) \to \nabla\psi$
A3	$\Delta\varphi \wedge \nabla(\varphi \vee \psi) \to \Delta(\neg\varphi \vee \chi)$
RΔ	If $\vdash \varphi$ then $\vdash \Delta\varphi$
RE	If $\vdash \varphi \leftrightarrow \psi$ then $\vdash \Delta\varphi \leftrightarrow \Delta\psi$
MP	If $\vdash \varphi$ and $\vdash \varphi \to \psi$ then $\vdash \psi$.

- $\mathbf{K4\Delta} = \mathbf{K\Delta} + \Delta\varphi \to \Delta(\Delta\varphi \vee \psi)$ = transitive non-contingency logic.

Zolin's logics

- $\sharp(x) = \{\varphi \mid \boxtimes\varphi \subseteq x\}$, where $\boxtimes\varphi = \{\Delta(\psi \to \varphi) \mid \psi \in \mathcal{L}(\Delta)\}$.
- $\mathbf{K}^{\mathbf{\Delta}}$

$(\mathbf{A}^{\mathbf{\Delta}}_{\top})$	All classical tautologies in $\mathcal{L}(\Delta)$
$(\mathbf{A}^{\mathbf{\Delta}}_{\mathbf{K}})$	$\Delta(\varphi \leftrightarrow \psi) \to (\Delta\varphi \leftrightarrow \Delta\psi)$
$(\mathbf{A}^{\mathbf{\Delta}}_{\neg})$	$\Delta\varphi \leftrightarrow \Delta\neg\varphi$
$(\mathbf{A}^{\mathbf{\Delta}}_{\vee})$	$\Delta\varphi \to [\Delta(\psi \to \varphi) \vee \Delta(\varphi \to \chi)]$
$(\mathbf{MP})$	
$(\mathbf{NCR})$	$\dfrac{\varphi}{\Delta\varphi}$

- $\mathbf{K4}^{\mathbf{\Delta}} = \mathbf{K}^{\mathbf{\Delta}} + \Delta\varphi \to \Delta(\psi \to \Delta\varphi)$
- $\mathbf{K5}^{\mathbf{\Delta}} = \mathbf{K}^{\mathbf{\Delta}} + \neg\Delta\varphi \to \Delta(\psi \to \neg\Delta\varphi)$
- NB: $\sharp$ is equal to Kuhn's λ

Limitations

- Humberstone [Humberstone, 1995]:
  $\lambda(s) = \{\varphi \mid \Delta\varphi \in s$ and for all $\psi, \vdash \varphi \to \psi$ implies $\Delta\psi \in s\}$
  is responsible for the infinitary axiomatization, and the completeness proof requires König's Lemma

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  $\lambda(s) = \{\varphi \mid \Delta\varphi \in s \text{ and for all } \psi, \vdash \varphi \to \psi \text{ implies } \Delta\psi \in s\}$
  is responsible for the infinitary axiomatization, and the completeness proof requires König's Lemma
- Kuhn [Kuhn, 1995] and Zolin [Zolin, 1999]:
  - The necessity operator, defined by $\boxtimes\varphi =_{df} \bigwedge_{\psi\in\mathcal{L}(\Delta)} \Delta(\varphi \vee \psi)$, is not really $\Box$. E.g., $\varphi \to \boxtimes\neg\boxtimes\neg\varphi$ is not valid on the class of symmetric frames [Zolin, 2001].
  - The canonical relations in [Kuhn, 1995, Zolin, 1999] at least do not apply to the reflexive frames, a fortiori, they do not apply to the symmetric frames [Humberstone, 2002a, page 118].

Almost-definability-schema based logics

[Fan et al., 2014, Fan et al., 2015]

- Almost-definability schema: Under a condition $\nabla\psi$ (viz. $\neg\Delta\psi$) for some ψ, $\Box$ is definable with Δ

$$\vDash \nabla\psi \to (\Box\varphi \leftrightarrow \Delta\varphi \wedge \Delta(\psi \to \varphi))$$

- $sR^c t$ iff there exists χ such that:
  - $\neg\Delta\chi \in s$, and
  - for all φ, $\Delta\varphi \wedge \Delta(\chi \to \varphi) \in s$ implies $\varphi \in t$.

- Standard modal logic: $sR^c t$ iff for all φ, $\Box\varphi \in s$ implies $\varphi \in t$

Axiomatization: the minimal logic

$\mathbb{NCL}$:

TAUT	all instances of tautologies
ΔCon	$\Delta(\chi \to \varphi) \wedge \Delta(\neg\chi \to \varphi) \to \Delta\varphi$
ΔDis	$\Delta\varphi \to \Delta(\varphi \to \psi) \vee \Delta(\neg\varphi \to \chi)$
ΔEqu	$\Delta\varphi \leftrightarrow \Delta\neg\varphi$
MP	From φ and $\varphi \to \psi$ infer ψ
NECΔ	From φ infer $\Delta\varphi$
REΔ	From $\varphi \leftrightarrow \psi$ infer $\Delta\varphi \leftrightarrow \Delta\psi$

NB: NECΔ is indispensable in $\mathbb{NCL}$

Axiomatization: extensions

Notation	Axiom Schemas	Systems
$\Delta\mathtt{T}$	$\Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \varphi \to \Delta\psi$	$\mathbb{NCLT} = \mathbb{NCL} + \Delta\mathtt{T}$
$\Delta 4$	$\Delta\varphi \to \Delta(\Delta\varphi \vee \psi)$	$\mathbb{NCL}4 = \mathbb{NCL} + \Delta 4$
$\Delta 5$	$\neg\Delta\varphi \to \Delta(\neg\Delta\varphi \vee \psi)$	$\mathbb{NCL}5 = \mathbb{NCL} + \Delta 5$
$\Delta\mathtt{B}$	$\varphi \to \Delta((\Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \neg\Delta\psi) \to \chi)$	$\mathbb{NCLB} = \mathbb{NCL} + \Delta\mathtt{B}$
$\mathtt{w}\Delta 4$	$\Delta\varphi \to \Delta\Delta\varphi$	$\mathbb{NCLS}4 = \mathbb{NCLT} + \mathtt{w}\Delta 4$
$\mathtt{w}\Delta 5$	$\neg\Delta\varphi \to \Delta\neg\Delta\varphi$	$\mathbb{NCLS}5 = \mathbb{NCLT} + \mathtt{w}\Delta 5$

- Completeness results w.r.t. corresponding classes of frames
- Apply to multimodal cases, except for that of $\mathbb{NCLB}$
- $\Delta\mathtt{T}$ and $\Delta\mathtt{B}$ can be obtained via almost-definability schema

$$\nabla\neg\psi \to (\Box\neg\varphi \to \neg\varphi) \quad (1)$$

$$\iff \nabla\neg\psi \wedge \Box\neg\varphi \to \neg\varphi \quad (2)$$

$$\iff \nabla\neg\psi \wedge \Delta\neg\varphi \wedge \Delta(\neg\psi \to \neg\varphi) \to \neg\varphi \quad (3)$$

$$\iff \Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \varphi \to \Delta\psi \quad (4)$$

Proof system for the symmetric frames

$\mathbb{NCLB}$: (NB: no need of the rule ($\texttt{NEC}\Delta$): $\dfrac{\varphi}{\Delta\varphi}$)

TAUT	all instances of tautologies
ΔCon	$\Delta(\chi \to \varphi) \wedge \Delta(\neg\chi \to \varphi) \to \Delta\varphi$
ΔDis	$\Delta\varphi \to \Delta(\varphi \to \psi) \vee \Delta(\neg\varphi \to \chi)$
ΔEqu	$\Delta\varphi \leftrightarrow \Delta\neg\varphi$
ΔB	$\varphi \to \Delta((\Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \neg\Delta\psi) \to \chi)$
MP	From φ and $\varphi \to \psi$ infer ψ
REΔ	From $\varphi \leftrightarrow \psi$ infer $\Delta\varphi \leftrightarrow \Delta\psi$

Proposition

$\mathbb{NCLB}$ *is sound with respect to the class of symmetric frames.*

Pseudo-Canonical Model

Definition (Pseudo-Canonical Model)

Define $\mathcal{M}^c = \langle S^c, R^c, V^c \rangle$ as follows:

- $S^c = \{s \mid s \text{ is a maximal consistent set of } \mathbb{NCLB}\}$
- For all $s, t \in S^c$, $sR^c t$ iff there exists χ such that:
  - $\neg\Delta\chi \in s$, and
  - for all φ, $\Delta\varphi \wedge \Delta(\chi \to \varphi) \in s$ implies $\varphi \in t$.
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

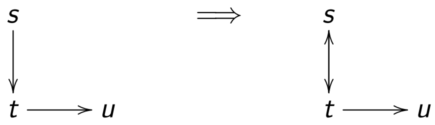
Lemma (Pseudo-Truth Lemma)

For all $\varphi \in \mathcal{L}(\Delta)$ and $s \in S^c$, $\mathcal{M}^c, s \vDash \varphi$ iff $\varphi \in s$.

R^c is *not* symmetric

Proposition

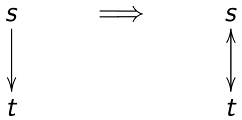
For any $s, t \in S^c$, if $sR^c t$ and $\neg\Delta\chi \in t$ for some χ, then $tR^c s$.



$sR^c t$ iff there exists χ such that:

- $\neg\Delta\chi \in s$, and
- for all φ, $\Delta\varphi \wedge \Delta(\chi \to \varphi) \in s$ implies $\varphi \in t$.

Turn $\mathcal{M}^c$ into a symmetric model



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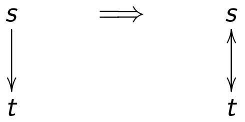
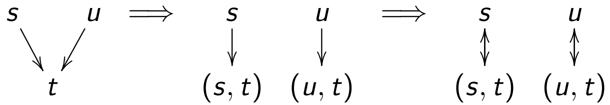


Split the world t:



Canonical Model of $\mathbb{NCLB}$

Definition

The canonical model $\mathcal{M}^+$ of $\mathbb{NCLB}$ is a tuple $\langle S^+, R^+, f, V^+\rangle$ where:

- $S^+ = \bar{D} \cup \{(s,t) \mid t \in D, sR^c t\}$
- sR^+t iff one of the following cases holds:
  1. $s, t \in \bar{D}$ and $sR^c t$,
  2. $s \in \bar{D}$ and $t = (s, s') \in S^+$,
  3. $t \in \bar{D}$ and $s = (t, t') \in S^+$.
- f is a function assigning each state in S^+ to a maximal consistent set in S^c such that $f(s) = s$ for $s \in \bar{D}$, and $f((s,t)) = t$ for $(s,t) \in S^+$.
- $V^+(p) = \{s \in S^+ \mid p \in f(s)\}$

where $D = \{t \mid t \in S^c, \Delta\chi \in t$ for all χ, and there exists an $s \in S^c$ such that $sR^c t\}$, where S^c and R^c are defined as in Definition 14, and $\bar{D} = S^c \setminus D$.

f acts like a surjective bounded morphism

Proposition

1. *f is surjective.*
2. *s and $f(s)$ satisfy the same propositional variables.*
3. *if $s \in \bar{D}$ then sR^+t implies $f(s)R^c f(t)$.*
4. *if $f(s)R^c t$ then there exists $u \in S^+$ such that $f(u) = t$ and sR^+u.*

$\mathcal{M}^+$ is desired canonical model

Lemma

$\mathcal{M}^+$ is symmetric.

Proposition

$\mathcal{M}^+$ preserves the truth values of formulas w.r.t. f. That is: for any $s \in S^+$ and any $\varphi \in \mathcal{L}(\Delta)$, we have

$$\mathcal{M}^+, s \vDash \varphi \iff \mathcal{M}^c, f(s) \vDash \varphi.$$

Completeness of $\mathbb{NCLB}$

Theorem

$\mathbb{NCLB}$ *is (sound and) strongly complete with respect to the class of symmetric frames.*

Multimodal $\mathcal{L}(\Delta)$

- Language ($i \in \mathbf{I}$, where $\mathbf{I}$ is finite)

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Delta_i\varphi \mid \Box_i\varphi$$

$\Box_i\varphi$: φ is necessary for agent i
$\Delta_i\varphi$: φ is non-contingent for agent i, i.e., for i, φ is necessarily true or φ is necessarily false.

- Semantics

$\mathcal{M}, s \vDash \Delta_i\varphi$	$\Leftrightarrow$	for any t_1, t_2 such that $s \to_i t_1, s \to_i t_2$: $(\mathcal{M}, t_1 \vDash \varphi \Leftrightarrow \mathcal{M}, t_2 \vDash \varphi)$

- Multimodal $\mathcal{L}(\Delta)$ is not normal
- Almost-definability

$$\vDash \nabla_i\psi \to (\Box_i\varphi \leftrightarrow \Delta_i\varphi \wedge \Delta_i(\psi \to \varphi))$$

Proof system for the symmetric frames: Multimodal case

$\mathbb{NCLB}_m$

TAUT	all instances of tautologies
ΔCon	$\Delta_i(\chi \to \varphi) \wedge \Delta_i(\neg\chi \to \varphi) \to \Delta_i\varphi$
ΔDis	$\Delta_i\varphi \to \Delta_i(\varphi \to \psi) \vee \Delta_i(\neg\varphi \to \chi)$
ΔEqu	$\Delta_i\varphi \leftrightarrow \Delta_i\neg\varphi$
ΔB	$\varphi \to \Delta_i((\Delta_i\varphi \wedge \Delta_i(\varphi \to \psi) \wedge \neg\Delta_i\psi) \to \chi)$
MP	From φ and $\varphi \to \psi$ infer ψ
REΔ	From $\varphi \leftrightarrow \psi$ infer $\Delta_i\varphi \leftrightarrow \Delta_i\psi$

Proposition

$\mathbb{NCLB}_m$ *is sound with respect to the class of symmetric frames.*

Pseudo-Canonical Model: again

Definition (Pseudo-Canonical Model)

Define $\mathcal{M}^c = \langle S^c, \{\rightarrow_i^c \mid i \in \mathbf{I}\}, V^c \rangle$ as follows:

- $S^c = \{s \mid s$ is a maximal consistent set of $\mathbb{NCLB}_m\}$
- For all $s, t \in S^c$, for all $i \in \mathbf{I}$, $s \rightarrow_i^c t$ iff there exists χ such that
  1. $\neg \Delta_i \chi \in s$, and
  2. for all φ, $\Delta_i \varphi \wedge \Delta_i(\chi \rightarrow \varphi) \in s$ implies $\varphi \in t$.
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

- Standard multi-modal logic: $s \rightarrow_i^c t$ iff for all φ, $\Box_i \varphi \in s$ implies $\varphi \in t$
- Almost-definability: $\neg \Delta_i \chi \rightarrow (\Box_i \varphi \leftrightarrow \Delta_i \varphi \wedge \Delta_i(\chi \rightarrow \varphi))$

Pseudo-Truth Lemma: again

Lemma

For all $\varphi \in \mathcal{L}(\Delta)$ *and* $s \in S^c$, $\mathcal{M}^c, s \models \varphi$ *iff* $\varphi \in s$.

$\rightarrow_i^c$ is *not* symmetric

Proposition

For any $s, t \in S^c$ and any $i \in \mathbf{I}$, if $s \rightarrow_i^c t$ and $t \rightarrow_i^c t'$ for some $t' \in S^c$, then $t \rightarrow_i^c s$.

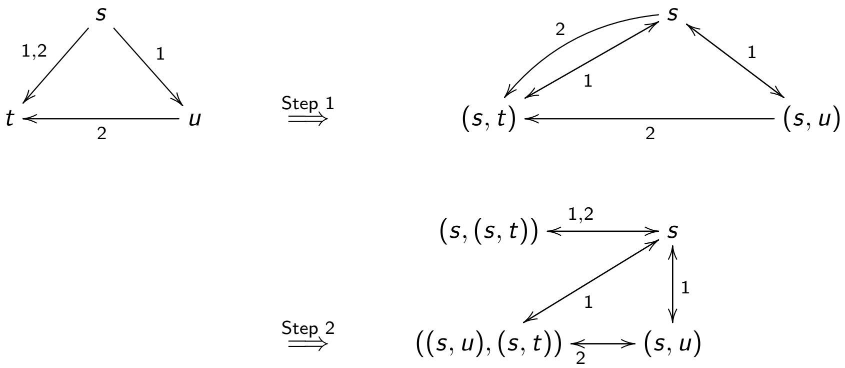
- The canonical model for $\mathbb{NCLB}$ cannot be generalized into $\mathbb{NCLB}_m$
- The dead ends are relative to the agents
- A dead end for agent j may be not a dead end for agent i
- Need new strategy

New Strategy: turn $\mathcal{M}^c$ into a symmetric model

Enumerate all of the agents in **I** as $1, 2, 3, \cdots, m$. Starting from $\mathcal{M}^0 = \mathcal{M}^c$ (we may as well assume that $\mathcal{M}^c$ has run out of Prop. 16), we construct the desired model (call it $\mathcal{M}^m$) in m steps.

- In each step we tackle the *dead ends* for that agent, by replacing those dead ends with some new copies of themselves such that each copy has only one incoming transition for that agent and then adding the back arrows for the agent
- while keeping all the arrows for the other agents in place, with corresponding replacements for the dead ends. We have to provide that
  1. In each step, the accessibility relation for that agent is symmetric,
  2. The symmetry of the previous relation for a fixed agent is not broken, which guarantee $\mathcal{M}^m$ to be symmetric
  3. Each step preserves the truth values of formulas

An example



Canonical model $\mathcal{M}^m$ of $\mathbb{NCLB}_m$

Definition

Define $\mathcal{M}^m = \langle S^m, \{\rightarrow_i^m \mid i \in \mathbf{I}\}, f^m, V^m \rangle$ by induction on $n \leq m$:

- $S^0 = S^c$
- $S^n = \bar{D}_n \cup \{(s, t) \mid t \in D_n \text{ and } s \rightarrow_n^{n-1} t\}$, where $D_n = \{t \mid t \in S^{n-1}, \text{ there is no } t' \in S^{n-1} \text{ such that } t \rightarrow_n^{n-1} t' \text{ and there exists an } s \in S^{n-1} \text{ such that } s \rightarrow_n^{n-1} t\}$, $\bar{D}_n = S^{n-1} \backslash D_n$

Canonical model $\mathcal{M}^m$ of $\mathbb{NCLB}_m$

Definition (Cont')

Define $\mathcal{M}^m = \langle S^m, \{\to_i^m \mid i \in \mathbf{I}\}, f^m, V^m \rangle$ by induction on $n \leq m$:

- $\to_n^0 = \to_n^c$
- $s \to_n^n t$ iff one of the following cases holds:
  1. $s, t \in \bar{D}_n$ and $s \to_n^{n-1} t$,
  2. $s \in \bar{D}_n$ and $t = (s, s') \in S^n$,
  3. $t \in \bar{D}_n$ and $s = (t, t') \in S^n$.
- For $i \neq n$, $s \to_i^n t$ iff one of the following cases holds:
  1. $s, t \in \bar{D}_n$ and $s \to_i^{n-1} t$,
  2. $s \in \bar{D}_n$ and $t = (s'', s') \in S^n$ and $s \to_i^{n-1} s'$,
  3. $t \in \bar{D}_n$ and $s = (t'', t') \in S^n$ and $t' \to_i^{n-1} t$,
  4. $s = (w, v) \in S^n$ and $t = (w', v') \in S^n$ and $v \to_i^{n-1} v'$.

Canonical model $\mathcal{M}^m$ of $\mathbb{NCLB}_m$

Definition (Cont')

Define $\mathcal{M}^m = \langle S^m, \{\rightarrow_i^m \mid i \in \mathbf{I}\}, f^m, V^m \rangle$ by induction on $n \leq m$:

- f^{n+1} is a function from S^{n+1} to S^n such that $f^{n+1}(s) = s$ for $s \in \bar{D}_{n+1}$, and $f^{n+1}((s,t)) = t$ for $(s,t) \in S^{n+1}$
- $V^0(p) = \{s \in S^c \mid p \in s\}$ and $V^{n+1}(p) = \{s \in S^{n+1} \mid f^{n+1}(s) \in V^n(p)\}$

Properties of f^{n+1}

Proposition (Preservation)

Given any $s, t \in S^{n+1}$. If $f^{n+1}(s) \to_i^n f^{n+1}(t)$, then

1. *If $i \neq n+1$, then $s \to_i^{n+1} t$.*
2. *If $i = n+1$, then for some $t' \in S^{n+1}$ such that $s \to_i^{n+1} t'$ and $f^{n+1}(t) = f^{n+1}(t')$.*

Proposition (No Miracle)

Given any $s, t \in S^{n+1}$.

1. *If $i \neq n+1$, then $s \to_i^{n+1} t$ implies $f^{n+1}(s) \to_i^n f^{n+1}(t)$.*
2. *If $i = n+1$ and $s \in \bar{D}_{n+1}$, then $s \to_i^{n+1} t$ implies $f^{n+1}(s) \to_i^n f^{n+1}(t)$.*

For every $n \in [0, m-1]$, f^{n+1} is surjective!

$\mathcal{M}^m$ is symmetric

Proposition

$\mathcal{M}^m$ is symmetric. That is, for all $i \in [1, m]$, $\rightarrow_i^m$ is symmetric:

1. *For every $n \in [1, m]$, $\rightarrow_n^n$ is symmetric.*
2. *If $\rightarrow_i^n$ is symmetric, then $\rightarrow_i^{n+1}$ is also symmetric.*

Truth-preserving in each step

Proposition

For any $n \in [0, m-1]$, *any* $s \in S^{n+1}$, *and any* $\varphi \in \mathcal{L}(\Delta)$,

$$\mathcal{M}^{n+1}, s \vDash \varphi \Longleftrightarrow \mathcal{M}^{n}, f^{n+1}(s) \vDash \varphi.$$

Completeness of $\mathbb{NCLB}_m$

Define $f = f^1 \circ f^2 \circ \cdots \circ f^m$.

- $f : S^m \to S^0$ is surjective.
- For any $s \in S^m$ and any $\varphi \in \mathcal{L}(\Delta)$, we have

$$\mathcal{M}^m, s \vDash \varphi \Longleftrightarrow \varphi \in f(s)$$

Theorem

$\mathbb{NCLB}_m$ *is strongly complete with respect to the class of symmetric frames.*

3. Neighborhood semantics and Axiomatizations

Neighborhood properties

$\mathcal{M} = \langle S, N, V \rangle$ is a *neighborhood model*, if S is a nonempty set of states, $N : S \to \mathcal{P}(\mathcal{P}(S))$ is a neighborhood function, and V is a valuation.

Definition (Neighborhood properties)

(n): $N(s)$ *contains the unit*, if $S \in N(s)$.
(i): $N(s)$ *is closed under intersections*, if $X, Y \in N(s)$ implies $X \cap Y \in N(s)$.
(s): $N(s)$ is *supplemented*, or *closed under supersets*, if $X \in N(s)$ and $X \subseteq Y \subseteq S$ implies $Y \in N(s)$.
(c): $N(s)$ is *closed under complements*, if $X \in N(s)$ implies $S \backslash X \in N(s)$.

A frame is called *quasi-filter*, if it possesses (i) and (s); a frame is called *filter*, if it has also (n).

Neighborhood semantics

[Fan and van Ditmarsch, 2015]

$\mathcal{M}, s \vDash p$	$\iff$	$s \in V(p)$
$\mathcal{M}, s \vDash \neg\varphi$	$\iff$	$\mathcal{M}, s \nvDash \varphi$
$\mathcal{M}, s \vDash \varphi \wedge \psi$	$\iff$	$\mathcal{M}, s \vDash \varphi$ and $\mathcal{M}, s \vDash \psi$
$\mathcal{M}, s \vDash \Delta\varphi$	$\iff$	$\varphi^{\mathcal{M}} \in N(s)$ or $S \backslash \varphi^{\mathcal{M}} \in N(s)$

TAUT	all instances of tautologies
ΔEqu	$\Delta\varphi \leftrightarrow \Delta\neg\varphi$
ΔM	$\Delta\varphi \to \Delta(\varphi \vee \psi) \vee \Delta(\neg\varphi \vee \chi)$
ΔC	$\Delta\varphi \wedge \Delta\psi \to \Delta(\varphi \wedge \psi)$
ΔN	$\Delta\top$
REΔ	$\dfrac{\varphi \leftrightarrow \psi}{\Delta\varphi \leftrightarrow \Delta\psi}$

systems	frame classes
$\mathbf{E}^{\boldsymbol\Delta} = \mathrm{TAUT} + \Delta\mathrm{Equ} + \mathrm{RE}\Delta$	all
$\mathbf{M}^{\boldsymbol\Delta} = \mathbf{E}^{\boldsymbol\Delta} + \Delta\mathrm{M}$	(s)
$(\mathbf{EC})^{\boldsymbol\Delta} = \mathbf{E}^{\boldsymbol\Delta} + \Delta\mathrm{C}$	$(i)\&(c)$
$(\mathbf{EN})^{\boldsymbol\Delta} = \mathbf{E}^{\boldsymbol\Delta} + \Delta\mathrm{N}$	(n)
$\mathbf{R}^{\boldsymbol\Delta} = \mathbf{M}^{\boldsymbol\Delta} + \Delta\mathrm{C}$	quasi-filters
$(\mathbf{EMN})^{\boldsymbol\Delta} = \mathbf{M}^{\boldsymbol\Delta} + \Delta\mathrm{N}$	$(s)\&(n)$
$(\mathbf{ECN})^{\boldsymbol\Delta} = (\mathbf{EC})^{\boldsymbol\Delta} + \Delta\mathrm{N}$	$(i)\&(c)\&(n)$
$\mathbf{K}^{\boldsymbol\Delta} = \mathbf{R}^{\boldsymbol\Delta} + \Delta\mathrm{N}$	filters

Definition

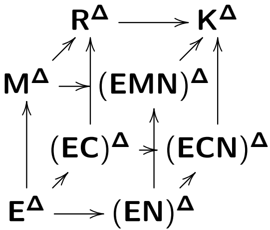
Let Σ be a system excluding ΔM. A tuple $\mathcal{M}^c = \langle S^c, N^c, V^c \rangle$ is a *canonical neighborhood model* for Σ, if

- $S^c = \{s \mid s \text{ is a maximal consistent set for } \Sigma\}$,
- $N^c(s) = \{|\varphi| \mid \Delta\varphi \in s\}$,
- $V^c(p) = |p|$.

Definition

Let Γ be a system including ΔM. A triple $\mathcal{M}^c = \langle S^c, N^c, V^c \rangle$ is a *canonical model* for Γ, if

- $S^c = \{s \mid s \text{ is a maximal consistent set for } \Gamma\}$,
- $N^c(s) = \{|\varphi| \mid \Delta(\varphi \vee \psi) \in s \text{ for every } \psi\}$ (Inspired by Kuhn's function λ: $\lambda(s) = \{\varphi \mid \forall\psi, \Delta(\varphi \vee \psi) \in s\}$.),
- For each $p \in \mathbf{P}$, $V^c(p) = |p|$.



- [Bakhtiari et al., 2017, p. 62] and [Bakhtiarinoodeh, 2017, pp. 124–125] claimed: "This raises the questions of what the axiomatizations are of monotone contingency logic and regular contingency logic. $\cdots$ one cannot fill these gaps with the axioms $\Delta\varphi \to \Delta(\varphi \to \psi) \vee \Delta(\neg\varphi \to \chi)$ and $\Delta(\psi \to \varphi) \wedge \Delta(\neg\psi \to \varphi) \to \Delta\varphi$. So these questions remain open."
- In $\mathbf{M}^\Delta$, ΔM by $\Delta\varphi \to \Delta(\varphi \to \psi) \vee \Delta(\neg\varphi \to \chi)$. In $\mathbf{R}^\Delta$, ΔC by $\Delta(\psi \to \varphi) \wedge \Delta(\neg\psi \to \varphi) \to \Delta\varphi$.
- The claim was wrong, and the two open questions are answered.

Reflection: how does the function λ arise?

- Kuhn's λ is very important for the definition of canonical relation and thus for the completeness proof in [Kuhn, 1995].
- It is this function that helps find simple axiomatizations for the minimal contingency logic and transitive contingency logic under Kripke semantics, so to speak.
- Despite its importance, the author did not say any intuitive idea about λ. And this function was thought of as 'ingenious' creation by some other researchers, say Humberstone [Humberstone, 2002b, p. 118] and Fan, Wang and van Ditmarch [Fan et al., 2015, p. 101].

Kuhn's λ is equal to Humberstone's λ

$$\lambda_H(s) = \{\varphi \mid \Delta\varphi \in s \text{ and } \forall\psi \text{ such that } \vdash \varphi \to \psi, \Delta\psi \in s\}.$$
$$\lambda_K(s) = \{\varphi \mid \forall\psi, \Delta(\varphi \vee \psi) \in s\}.$$

Proof.

- $\lambda_H(s) = \{\varphi \mid \forall\psi \text{ such that } \vdash \varphi \to \psi, \Delta\psi \in s\}$.
- Given the rule REΔ, (1) $\Longleftrightarrow$ (2):
  (1) For every ψ such that $\vdash \varphi \to \psi$, $\Delta\psi \in s$.
  (2) For every ψ, $\Delta(\varphi \vee \psi) \in s$.

□

Jie Fan. A sequence of neighborhood contingency logics. *arXiv preprint arXiv:1802.03516*, under submission, 2018.

Open questions

- Kripke semantics: axiomatizations of $\mathcal{L}(\Delta)$ over $B4$-frames
- Neighborhood semantics: proper extensions of $\mathbf{K}^{\mathbf{\Delta}}$

Thank you for your attention!

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