An Overview of Cardinals without the Axiom of Choice

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Introduction

Question

How to prove in ZF (without AC) that for all non-zero natural numbers n and all sets A, B, if $n \times A \approx n \times B$, then $A \approx B$?

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History of this question

- (Bernstein 1901) $2 \times A \approx 2 \times B \to A \approx B$
- (Sierpiński 1922) A simpler proof of $2 \times A \approx 2 \times B \to A \approx B$
- (Lindenbaum and Tarski 1926) Announcing the general case
- (Sierpiński 1947) $2 \times A \preccurlyeq 2 \times B \to A \preccurlyeq B$
- (Tarski 1949) $n \times A \preccurlyeq n \times B \to A \preccurlyeq B$
- (Doyle and Conway 1994) A new proof of $n \times A \preccurlyeq n \times B \to A \preccurlyeq B$

Introduction

Question

How to prove in ZF (without AC) that for all non-zero natural numbers n and all sets A, B, if $n \times A \approx n \times B$, then $A \approx B$?

Where is the difficulty?

- In the case where A or B is finite, we prove in ZF that $n \times A \approx n \times B \to A \approx B$ by invoking a bijection from A or B onto a natural number.
- In the case where A and B are infinite, we prove in ZFC that $n \times A \approx n \times B \to A \approx B$ by invoking a bijection from A or B onto an infinite (well-ordered) cardinal.
- Even in ZFC, it is difficult to define a bijection from A onto B by using only a bijection from $n \times A$ onto $n \times B$.

Preliminaries

Convention

Let $\varphi(p_1, \ldots, p_m, x_0, \ldots, x_n)$ and $\psi(p_1, \ldots, p_m, x_0, \ldots, x_n, y)$ be formulas in the language of set theory with no free variables other than indicated. When we say that *from* $x_0, \ldots, x_n$ *such that* $\varphi(p_1, \ldots, p_m, x_0, \ldots, x_n)$, *one can explicitly define a* y *such that* $\psi(p_1, \ldots, p_m, x_0, \ldots, x_n, y)$, we mean the following:

> *There exists a class function* G *without free variables such that if* $\varphi(p_1, \ldots, p_m, x_0, \ldots, x_n)$, *then* $(x_0, \ldots, x_n)$ *is in the domain of* G *and* $\psi(p_1, \ldots, p_m, x_0, \ldots, x_n, G(x_0, \ldots, x_n))$.

Preliminaries

Examples

- From a surjection $f\colon y \twoheadrightarrow x$ and a well-ordering r of y, one can explicitly define a well-ordering s of x.

  > *There exists a class function G without free variables such that if f is a surjection from y onto x and r well-orders y, then $G(f, r)$ is defined and is a well-ordering of x.*

- (Cantor-Bernstein) From an injection $f\colon x \rightarrow y$ and an injection $g : y \rightarrow x$, one can explicitly define a bijection $h : x \twoheadrightarrow y$.

  > *There exists a class function G without free variables such that if f is an injection from x into y and g is an injection from y into x, then $G(f, g)$ is defined and is a bijection from x onto y.*

Preliminaries

Project

Restate all theorems of ZFC in this form!

Preliminaries

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Further examples

- (Zermelo 1904) From a choice function on $\wp(x)$, one can explicitly define a well-ordering on x.
- (Faferman 1965) Even in ZFC, one cannot explicitly define a well-ordering of $\mathbb{R}$.
- (Jensen 1968) From a $\diamondsuit$-sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ and a ladder system $\langle C_\alpha \mid \alpha < \omega_1 \rangle$, one can explicitly define a Souslin tree.

Preliminaries

Definition of cardinality in ZF

$$|x| = \begin{cases} \min\{\alpha \mid \alpha \approx x\}, & \text{if } x \text{ is well-orderable;} \\ \{y \mid y \approx x \wedge \forall z \approx x (\mathrm{rank}(y) \leqslant \mathrm{rank}(z))\}, & \text{otherwise.} \end{cases}$$

We shall use lower case German letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ for cardinals.

Preliminaries

Definition

- $|x| + |y| = |x \times \{0\} \cup y \times \{1\}|$
- $|x| \cdot |y| = |x \times y|$
- $|y|^{|x|} = |\{f \mid f : x \to y\}|$

Preliminaries

Definition

- $x \preccurlyeq y$ means that there is an injection from x into y.
- $x \preccurlyeq^* y$ means that there is a surjection from a subset of y onto x.
- $\mathfrak{a} \leqslant \mathfrak{b}$ means that there are sets x, y such that $|x| = \mathfrak{a}$, $|y| = \mathfrak{b}$, and $x \preccurlyeq y$.
- $\mathfrak{a} \leqslant^* \mathfrak{b}$ means that there are sets x, y such that $|x| = \mathfrak{a}$, $|y| = \mathfrak{b}$, and $x \preccurlyeq^* y$.

Fact

$\mathfrak{a} \leqslant \mathfrak{b} \rightarrow \mathfrak{a} \leqslant^* \mathfrak{b} \rightarrow 2^{\mathfrak{a}} \leqslant 2^{\mathfrak{b}}$.

Preliminaries

If ZF is consistent, we cannot prove in ZF that every infinite set includes a denumerable subset, and we cannot even prove in ZF that the power set of an infinite set includes a denumerable subset. This suggests us to introduce the following definition.

Definition

- x is Dedekind infinite if $\omega \preccurlyeq x$; otherwise x is Dedekind finite.
- x is power Dedekind infinite if $\omega \preccurlyeq \wp(x)$; otherwise x is power Dedekind finite.
- $\mathfrak{a}$ is Dedekind infinite if $\aleph_0 \leqslant \mathfrak{a}$; otherwise $\mathfrak{a}$ is Dedekind finite.
- $\mathfrak{a}$ is power Dedekind infinite if $\aleph_0 \leqslant 2^{\mathfrak{a}}$; otherwise $\mathfrak{a}$ is power Dedekind finite.

Preliminaries

Fact

- $\mathfrak{a}$ is Dedekind infinite $\to$ $\mathfrak{a}$ is power Dedekind infinite $\to$ $\mathfrak{a}$ is infinite
- ZF $\nvdash$ $\mathfrak{a}$ is infinite $\to$ $\mathfrak{a}$ is power Dedekind infinite
- ZF $\nvdash$ $\mathfrak{a}$ is power Dedekind infinite $\to$ $\mathfrak{a}$ is Dedekind infinite
- (Dedekind 1888) $\mathfrak{a}$ is Dedekind infinite $\leftrightarrow$ $\mathfrak{a} + 1 = \mathfrak{a}$
- The class of all Dedekind finite sets is closed under disjoint unions.
- $\mathfrak{a}$ is infinite $\to$ $2^{\mathfrak{a}}$ is power Dedekind infinite

Preliminaries

Theorem (Kuratowski 1920s)

$\mathfrak{a}$ *is power Dedekind infinite* $\leftrightarrow \aleph_0 \leqslant^* \mathfrak{a} \leftrightarrow 2^{\aleph_0} \leqslant 2^{\mathfrak{a}}$

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Proof.

- From an infinite subset x of $\wp(\omega)$, one can explicitly define an infinite proper subset y of x.
- From an infinite subset x of $\wp(\omega)$, one can explicitly define a surjection $f\colon x \twoheadrightarrow \omega$.
- From an injection $f\colon \omega \to \wp(x)$, one can explicitly define a surjection $f\colon x \twoheadrightarrow \omega$.

$\square$

Preliminaries

Theorem (Kuratowski 1920s)

$\mathfrak{a}$ *is power Dedekind infinite* $\leftrightarrow \aleph_0 \leqslant^* \mathfrak{a} \leftrightarrow 2^{\aleph_0} \leqslant 2^{\mathfrak{a}}$

Corollary

The class of all power Dedekind finite sets is closed under unions.

Lindenbaum and Tarski's Theorem

Further results

- (Truss 1972) ZF $\nvdash 2 \times A \preccurlyeq^* 2 \times B \rightarrow A \preccurlyeq^* B$
- (Truss 1984)
  $n \times A \preccurlyeq^* n \times B \wedge n \times B \preccurlyeq^* n \times A \rightarrow A \preccurlyeq^* B \wedge B \preccurlyeq^* A$

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Problem

Is it provable in ZF that for all non-void power Dedekind finite sets d and all sets A, B, if $d \times A \approx d \times B$, then $A \approx B$?

Generalizations of Cantor's Theorem

Theorem (Cantor 1892)

$2^{\mathfrak{a}} \not\leq^* \mathfrak{a}$.

Moreover, from a function $f : x \to \wp(x)$, *one can explicitly define a* $u \in \wp(x) - \mathrm{ran}(f)$.

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Proof.

Let $u = \{z \in \mathrm{dom}(f) \mid z \notin f(z)\}$. □

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Remark

Note that $2^{\mathfrak{a}} \not\leq^* \mathfrak{a}$ is a consequence of the theorem that for all cardinals $\mathfrak{a}$, $2^{\mathfrak{a}} \not\leq \mathfrak{a}$: from $2^{2^{\mathfrak{a}}} \not\leq 2^{\mathfrak{a}}$, we get $2^{\mathfrak{a}} \not\leq^* \mathfrak{a}$.

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Theorem (Specker 1954)

For all infinite cardinals $\mathfrak{a}$, $2^{\mathfrak{a}} \nleq \mathfrak{a}^2$.

Generalizations of Cantor's Theorem

Theorem (Specker 1954)

For all infinite cardinals $\mathfrak{a}$, $2^{\mathfrak{a}} \not\leq \mathfrak{a}^2$.

Proof.

- From an infinite ordinal α, one can explicitly define an injection $f : \alpha \times \alpha \to \alpha$.
- From an injection $f : \alpha \to y \times y$, where α is an infinite ordinal, one can explicitly define an injection $g : \alpha \to y$.
- From an injection f from a subset of $\wp(y)$ into $y \times y$ and an injection $g : \omega \to \wp(y)$, one can explicitly define a $u \in \wp(y) - \mathrm{dom}(f)$.

□

Generalizations of Cantor's Theorem

Further results

- (Tarski 1939) $\mathrm{s}(x) \not\preccurlyeq x$; $\mathrm{s}(x) = \{y \subseteq x \mid y \text{ is well-orderable}\}$.
- (Truss 1973) For all infinite sets x, $\mathrm{s}(x) \not\preccurlyeq x^n$ and $\mathrm{w}(x) \not\preccurlyeq x^n$; $\mathrm{w}(x) = \{f \mid f \text{ is an injection from some ordinal into } x\}$.
- (Halbeisen and Shelah 1994) For all infinite sets x, $\wp(x) \not\preccurlyeq \mathrm{fin}(x)$, where $\mathrm{fin}(x) = \{y \subseteq x \mid y \text{ is finite}\}$.
- (Forster 2003) For all infinite sets x, there are no finite-to-one surjections from $\wp(x)$ onto x.
- (Vejjajiva and Panasawatwong 2014) For all power Dedekind infinite sets x, $\wp(x) \not\preccurlyeq \mathrm{pdfin}(x)$, where $\mathrm{pdfin}(x) = \{y \subseteq x \mid y \text{ is power Dedekind finite}\}$.
- (Keremedis 2016) It is consistent with ZF that there exists a Dedekind infinite set x such that $\wp(x) \preccurlyeq \mathrm{dfin}(x)$, where $\mathrm{dfin}(x) = \{y \subseteq x \mid y \text{ is Dedekind finite}\}$.

Generalizations of Cantor's Theorem

My work

- For all power Dedekind infinite sets x, $\wp(x) \not\preccurlyeq_{\mathrm{dfto}} \mathrm{pdfin}(x)$.
- For all sets x, if $\mathrm{s}(x)$ (resp., $\mathrm{w}(x)$) is Dedekind infinite, then $\mathrm{s}(x) \not\preccurlyeq_{\mathrm{dfto}} \mathrm{seq}^{\text{1-1}}(x)$ (resp., $\mathrm{w}(x) \not\preccurlyeq_{\mathrm{dfto}} \mathrm{seq}^{\text{1-1}}(x)$), where $\mathrm{seq}^{\text{1-1}}(x) = \{f \mid f \text{ is an injection from some } n \in \omega \text{ into } x\}$.
- It is consistent with ZF that there exists a Dedekind infinite set x such that $|\mathrm{w}(x)| < |[x]^2|$.
- For all sets x, y, if x is infinite and $y \preccurlyeq_{\mathrm{pdfto}} x$, then $\wp(x) \not\preccurlyeq^{*} y$.
- For all infinite sets x, $\wp(\mathrm{fin}(x)) \not\preccurlyeq^{*} \mathrm{seq}(\mathrm{fin}(x))$, where $\mathrm{seq}(y) = \{f \mid f \text{ is a function from some } n \in \omega \text{ into } x\}$.

Generalizations of Cantor's Theorem

The dual Specker problem

Is it provable in ZF that for all infinite cardinals $\mathfrak{a}$, $2^{\mathfrak{a}} \not\leq^* \mathfrak{a}^2$?

The dual Specker problem

Is it provable in ZF that for all infinite cardinals $\mathfrak{a}$, $2^{\mathfrak{a}} \not\leq^{*} \mathfrak{a}^2$?

Remark

Note that we have affirmatively answered a weaker version of this problem: if there exists an infinite cardinal $\mathfrak{b}$ such that $\mathfrak{a} = \mathrm{fin}(\mathfrak{b})$, then $2^{\mathfrak{a}} \not\leq^{*} \mathfrak{a}^2$.

GCH and AC

Definition

- AH (Aleph Hypothesis): $\forall\alpha(2^{\aleph_\alpha} = \aleph_{\alpha+1})$
- CH($\mathfrak{a}$): $\neg\exists\mathfrak{b}(\mathfrak{a} < \mathfrak{b} < 2^{\mathfrak{a}})$
- GCH: $\forall\mathfrak{a}(\mathfrak{a} < \omega \vee \mathrm{CH}(\mathfrak{a}))$

GCH and AC

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Theorem (H. Rubin 1960)

If for all well-ordered cardinals κ, $\wp(\kappa)$ is well-orderable, then AC.

Corollary

AH $\to$ AC

GCH and AC

GCH $\rightarrow$ AC

- (Lindenbaum and Tarski 1926) Announcing:
  $\mathrm{CH}(\mathfrak{a}) \wedge \mathrm{CH}(2^{\mathfrak{a}}) \wedge \mathrm{CH}(2^{2^{\mathfrak{a}}}) \rightarrow 2^{2^{\mathfrak{a}}}$ is a well-ordered cardinal;
  $\mathrm{CH}(\mathfrak{a}^2) \wedge \mathrm{CH}(2^{\mathfrak{a}^2}) \rightarrow 2^{\mathfrak{a}^2}$ is a well-ordered cardinal.
- (Sierpiński 1945)
  $\mathrm{CH}(\mathfrak{a}) \wedge \mathrm{CH}(2^{\mathfrak{a}}) \wedge \mathrm{CH}(2^{2^{\mathfrak{a}}}) \rightarrow \mathfrak{a}$ is a well-ordered cardinal.
- (Specker 1954)
  $\mathrm{CH}(\mathfrak{a}) \wedge \mathrm{CH}(2^{\mathfrak{a}}) \rightarrow 2^{\mathfrak{a}}$ is a well-ordered cardinal.
- (Kruse 1960, Kanamori and Pincus 2002) If $\mathrm{CH}(\mathfrak{a})$ and there are no increasing sequences of cardinals of length $\mathrm{cf}(\aleph(\mathfrak{a}))$ between $2^{\mathfrak{a}}$ and $2^{2^{\mathfrak{a}}}$, then $2^{\mathfrak{a}}$ is a well-ordered cardinal.
- (Kanamori and Pincus 2002)
  $\mathrm{ZF} \nvdash \mathrm{CH}(\mathfrak{a}) \rightarrow 2^{\mathfrak{a}}$ is a well-ordered cardinal

GCH and AC

Does GCH imply AC locally?

ZF $\vdash$ CH($\mathfrak{a}$) $\rightarrow$ $\mathfrak{a}$ is a well-ordered cardinal ?

Lauchli's Theorem

Theorem (Lauchli 1961)

For all infinite cardinals $\mathfrak{a}$, $2^{2^{\mathfrak{a}}} + 2^{2^{\mathfrak{a}}} = 2^{2^{\mathfrak{a}}}$.

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Fact

- $\mathfrak{a}$ is Dedekind finite $\to \mathfrak{a} + \mathfrak{a} > \mathfrak{a}$
- $\mathfrak{a}$ is power Dedekind finite $\to 2^{\mathfrak{a}} + 2^{\mathfrak{a}} > 2^{\mathfrak{a}}$

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Lemma (Lauchli 1961)

For all infinite cardinals $\mathfrak{a}$, $2^{\aleph_0 \cdot \mathrm{fin}(\mathfrak{a})} = 2^{\mathrm{fin}(\mathfrak{a})}$.

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For all infinite cardinals $\mathfrak{a}$, $2^{\aleph_0 \cdot \mathrm{fin}(\mathfrak{a})} = 2^{\mathrm{fin}(\mathfrak{a})}$.

Fact

- $\mathfrak{a}$ is Dedekind infinite $\rightarrow \aleph_0 \cdot \mathrm{fin}(\mathfrak{a}) = \mathrm{fin}(\mathfrak{a})$
- $\mathfrak{a}$ is power Dedekind infinite $\rightarrow \aleph_0 \cdot \mathrm{fin}(\mathfrak{a}) \leqslant^* \mathrm{fin}(\mathfrak{a})$
- (Truss 1974) ZF $\nvdash$ $\mathfrak{a}$ is infinite $\rightarrow \aleph_0 \cdot \mathrm{fin}(\mathfrak{a}) \leqslant^* \mathrm{fin}(\mathfrak{a})$

Lauchli's Theorem

Lemma (Lauchli 1961)

For all infinite cardinals $\mathfrak{a}$, $2^{\aleph_0 \cdot \mathrm{fin}(\mathfrak{a})} = 2^{\mathrm{fin}(\mathfrak{a})}$.

Proof

Let A be a fixed set. For all n, k such that $n \leqslant k$, we define:

- $F_{n,k} : \wp([A]^n) \to \wp([A]^k)$ such that for all $X \subseteq [A]^n$,

$$F_{n,k}(X) = \{y \in [A]^k \mid \exists x \in X(x \subseteq y)\}$$

- $G_{n,k} : \wp([A]^n) \to \wp([A]^n)$ such that for all $X \subseteq [A]^n$,

$$G_{n,k}(X) = \{x \in [A]^n \mid \forall y \in [A]^k(x \subseteq y \to y \in F_{n,k}(X))\}$$

- For all $X \subseteq [A]^n$, $H_{n,k}(X) = G_{n,k}(X) - X$.

Lauchli's Theorem

Fact

1. $X \subseteq Y \subseteq [A]^n \to F_{n,k}(X) \subseteq F_{n,k}(Y)$
2. $X \subseteq [A]^n \to X \subseteq G_{n,k}(X)$
3. $X \subseteq Y \subseteq [A]^n \to G_{n,k}(X) \subseteq G_{n,k}(Y)$
4. $X \subseteq [A]^n \to G_{n,k}(G_{n,k}(X)) = G_{n,k}(X)$
5. $X \subseteq [A]^n \to F_{n,k}(G_{n,k}(X)) = F_{n,k}(X)$
6. $F_{n,k} \restriction \{X \subseteq [A]^n \mid G_{n,k}(X) = X\}$ is 1-1.
7. For all $X \subseteq [A]^n$ and all natural numbers m,

$$H^m_{n,k}(X) = G_{n,k}(H^m_{n,k}(X)) - H^{m+1}_{n,k}(X)$$

8. $k \leqslant k' \wedge X \subseteq [A]^n \to G_{n,k}(X) \subseteq G_{n,k'}(X)$, and hence

$$\{X \subseteq [A]^n \mid G_{n,k'}(X) = X\} \subseteq \{X \subseteq [A]^n \mid G_{n,k}(X) = X\}$$

Lauchli's Theorem

Key Lemma

$X \subseteq [A]^n \rightarrow H^{n+1}_{n,k}(X) = \varnothing$

Corollary

$X \subseteq [A]^n \rightarrow H^{n}_{n,k}(X) = G_{n,k}(H^{n}_{n,k}(X))$

Lauchli's Theorem

Come back to the proof of Lauchli's Lemma

For all $X \subseteq \omega \times \operatorname{fin}(A)$ and all natural numbers i, n, m, we define:

$$X^{(0)}_{i,n} = X``\{i\} \cap [A]^n$$
$$X^{(1)}_{i,n,m} = G_{n,2^i3^n5^n}(H^m_{n,2^i3^n5^n}(X^{(0)}_{i,n}))$$
$$X^{(2)}_{i,n,m} = F_{n,2^i3^n5^m}(X^{(1)}_{i,n,m})$$

Let

$$\Phi(X) = \bigcup_{i\in\omega} \bigcup_{n\in\omega} \bigcup_{m=0}^{n} X^{(2)}_{i,n,m}$$

Lauchli's Theorem

Note that if $m \leqslant n$, then

- $X^{(2)}_{i,n,m} = \Phi(X) \cap [A]^{2^i 3^n 5^m}$
- $X^{(1)}_{i,n,m} = (F_{n,2^i 3^n 5^m} \upharpoonright \{Y \subseteq [A]^n \mid G_{n,2^i 3^n 5^m}(Y) = Y\})^{-1}(X^{(2)}_{i,n,m})$
- $X^{(0)}_{i,n} = X^{(1)}_{i,n,0} - (X^{(1)}_{i,n,1} - (\cdots(X^{(1)}_{i,n,n-1} - X^{(1)}_{i,n,n})\cdots))$
- $X = \bigcup\{\{i\} \times X^{(0)}_{i,n} \mid i, n \in \omega\}$

Hence, Φ is an injection from $\wp(\omega \times \mathrm{fin}(A))$ into $\wp(\mathrm{fin}(A))$. $\square$

Lauchli's Theorem

My work

- For all infinite cardinals $\mathfrak{a}$, $2^{(\mathrm{fin}(\mathfrak{a}))^n} = 2^{[\mathrm{fin}(\mathfrak{a})]^n}$.
- For all infinite cardinals $\mathfrak{a}$ and all $m > 1$,

$$2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))} = 2^{\mathrm{fin}^m(\mathfrak{a})} = 2^{\mathrm{seq}(\mathfrak{a})} = 2^{\mathrm{seq}(\mathrm{fin}(\mathfrak{a}))} = 2^{\mathrm{seq}(\mathrm{seq}(\mathfrak{a}))}$$

Lauchli's Theorem

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Problems

- $\mathsf{ZF} \vdash \mathfrak{a}$ is infinite $\rightarrow 2^{\mathrm{fin}(\mathfrak{a})} = 2^{\mathrm{fin}(\mathrm{fin}(\mathfrak{a}))}$?
- $\mathsf{ZF} \vdash \mathfrak{a}$ is infinite $\rightarrow 2^{2^{\mathfrak{a}}} \cdot 2^{2^{\mathfrak{a}}} = 2^{2^{\mathfrak{a}}}$?

Thank you