Borel Functions and Separability of Metric Spaces

Delta 8 Logic Workshop (Zhejiang University)

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Outline

Question

Theorem (S. Simpson)

Let X be an analytic set, Y a metric space, and $f\colon X \to Y$ a Borel function. Then $f(X)$ is separable.

Question: If X is only a separable metric space, is $f(X)$ separable?

We prove that this problem is independent of ZFC.

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Borel sets

Let X be a metric space.

Definition

$\mathbf{B}(X)$: *Borel sets* of X is the σ-algebra generated by the open sets of X.

Borel hierarchy

$$\mathbf{\Sigma}^0_1 = \text{open}, \quad \mathbf{\Pi}^0_1 = \text{closed};$$

For $1 < \alpha < \omega_1$,

$$\mathbf{\Sigma}^0_\alpha = \{\bigcup_{n \in \omega} A_n : A_n \in \mathbf{\Pi}^0_{\alpha_n}, \alpha_n < \alpha\};$$

$$\mathbf{\Pi}^0_\alpha = \{X \setminus A : A \in \mathbf{\Sigma}^0_\alpha(X)\}$$

$$\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha.$$

$$\mathbf{B}(X) = \bigcup_{1 \leq \alpha < \omega_1} \mathbf{\Sigma}^0_\alpha(X) = \bigcup_{1 \leq \alpha < \omega_1} \mathbf{\Pi}^0_\alpha(X).$$

Borel functions

Let X and Y be metric spaces, and let $f : X \to Y$ be a function.

Definition

Borel function: $f^{-1}(U)$ is Borel set in X for any U open in Y.

Polish spaces

Definition

Polish space: a separable, completely metrizable topological space.

Example

1. $\mathbb{R}^n, \mathbb{C}^n$ and $\mathbb{I} = [0, 1]$;
2. countable discrete spaces;
3. products of countable many Polish spaces:
   (a) Hilbert cube $\mathbb{I}^\omega$,
   (b) Cantor space $\mathcal{C} = \{0, 1\}^\omega$,
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Analytic sets

Definition

Let X be a Polish space. A subset $A \subseteq X$ is *analytic* (or $\mathbf{\Sigma}^1_1$) if there is a closed subset $C \subseteq X \times \mathcal{N}$ such that

$$x \in A \iff \exists y \in \mathcal{N}((x,y) \in C).$$

A subset $B \subseteq X$ is *co-analytic* (or $\mathbf{\Pi}^1_1$) if $X \setminus B$ is analytic.

Theorem (Suslin)

Let X be a Polish space and $A \subseteq X$. Then A is Borel iff both A and $X \setminus A$ are analytic.

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Projective sets

Let X be a Polish space. We have already defined the $\mathbf{\Sigma}^1_1$ (analytic), $\mathbf{\Pi}^1_1$ (co-analytic) sets.

$$\mathbf{\Sigma}^1_{n+1} = \{\mathrm{proj}_X(A) : X \text{ Polish}, A \subseteq X \times \mathcal{N}, A \in \mathbf{\Pi}^1_n(X \times \mathcal{N})\}$$

$$\mathbf{\Pi}^1_{n+1} = \{X \setminus A : X \text{ Polish}, A \in \mathbf{\Sigma}^1_{n+1}(X)\}$$

$$\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$$

$$\mathbf{P}(X) = \bigcup_{n\in\omega} \mathbf{\Sigma}^1_n(X) = \bigcup_{n\in\omega} \mathbf{\Pi}^1_n(X)$$

Determinacy

With each subset A of ω^ω we associate the following game G_A, played by two players I and II. First I chooses a natural number a_0, then II chooses a natural number b_0, then I chooses a_1, then II chooses b_1, and so on. The game ends after ω steps; if the resulting sequence $\langle a_0, b_0, a_1, b_1, ...\rangle$ is in A, then I wins, otherwise II wins.

A *strategy* (for I or II) is a rule that tells the player what move to make depending on the previous moves of both players. A strategy is a *winning strategy* if the player who follows it always wins. The game G_A is *determined* if one of the players has a winning strategy.

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Theorem (Martin, 1975)

All Borel games are determined.

Definition

Projective Determinacy (PD): the game G_A is determined for every projective set A.

Theorem (Martin)

If Determinacy $(\mathbf{\Pi}^1_n)$ *then every uncountable* $\mathbf{\Sigma}^1_{n+1}$ *subset of* $\{0,1\}^\omega$ *has a perfect subset.*

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Martin's Axiom (MA)

Let $\langle \mathbb{P}, \leq \rangle$ be a partial order. A *chain* in $\mathbb{P}$ is a set $C \subseteq \mathbb{P}$ such that $\forall p, q \in C(p \leq q \vee q \leq p)$. p and q are *compatible* iff

$$\exists r \in \mathbb{P}(r \leq p \wedge r \leq q);$$

they are *incompatible* ($p \perp q$) iff $\neg \exists r \in \mathbb{P}(r \leq p \wedge r \leq q)$. An *antichain* in $\mathbb{P}$ is a subset $A \subseteq \mathbb{P}$ such that $\forall p, q \in A(p \neq q \rightarrow p \perp q)$.

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A partial order $\langle \mathbb{P}, \leq \rangle$ has the *countable chain condition* (c.c.c.) iff every antichain in $\mathbb{P}$ is countable.

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Let $\langle \mathbb{P}, \leq \rangle$ be a partial order. $D \subseteq \mathbb{P}$ is *dense* in $\mathbb{P}$ iff $\forall p \in \mathbb{P} \exists q \leq p (q \in D)$. $G \subseteq \mathbb{P}$ is a *filter* in $\mathbb{P}$ iff

(a) $\forall p, q \in G \exists r \in G (r \leq p \wedge r \leq q)$, and

(b) $\forall p \in \mathbb{P} \forall q \in G (q \leq p \rightarrow p \in G)$.

MA(κ) is the statement: Whenever $\langle \mathbb{P}, \leq \rangle$ is a non-empty c.c.c. partial order, and $\mathscr{D}$ is a family of $\leq \kappa$ dense subsets of $\mathbb{P}$, then there is a filter G in $\mathbb{P}$ such that $\forall D \in \mathscr{D} (G \bigcap D \neq \varnothing)$. MA is the statement $\forall \kappa < 2^\omega$ (MA(κ)).

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Outline

Recall the question

Let X and Y be metric spaces with X separable, and let $f : X \to Y$ be a Borel function. Is then $f(X)$ separable?

Theorems

We prove that this problem is independent of ZFC due to the following theorems:

Theorem ($2^{\omega_1} > 2^{\omega}$)

Let X and Y be metric spaces with X separable, and let $f : X \to Y$ be a Borel function. Then $f(X)$ is separable.

Theorem (MA(ω_1))

There exist metric spaces X and Y with X separable, and a Borel function $f : X \to Y$ such that $f(X)$ is not separable. Furthermore, here f can be of Baire class 1.

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Theorems

Theorem (MA $+ \neg$CH $+ \omega_1 = \omega_1^L$)

There exist metric spaces X and Y with X a co-analytic subset of $\mathbb{R}$, and a Borel function $f : X \to Y$ such that $f(X)$ is not separable. Furthermore, here f can be of Baire class 1.

Theorems

Theorem (PD)

Let X be a projective set, Y a metric space, and $f : X \to Y$ a Borel function. Then $f(X)$ is separable.

Proof

Proof.

Suppose $f(X)$ is not separable, there is a closed discrete subspace Z of $f(X)$ with $|Z| = \omega_1$. Let $W = f^{-1}(Z)$. It is easy to see that W is a Borel set in X, so W is a projective set. Now we take a Bernstein set B of $\mathbb{R}$. Let $A \subseteq B$ with $|A| = |Z| = \omega_1$. Then A does not contain any uncountable closed set. Let g be an one-to-one map from Z onto A. Since f is Borel and g is continuous, $g \circ f \upharpoonright W$ is a Borel function. Since $A = g(f(W))$ which is the image of a projective set under a Borel function, A is a projective set. Then A is an uncountable projective set containing no perfect set, this contradicts with PD. □

Corollary

Definition

Let X and Y be metric spaces, and let $f : X \to Y$ be a Borel function. The Borel function f is *bounded* if there is an ordinal α, $1 \leq \alpha < \omega_1$, such that $f^{-1}(U) \in \Sigma^0_\alpha$ for every open set U of Y. Otherwise we call f *unbounded*.

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An open problem

An open problem which was put forward by A. H. Stone: let X and Y be metric spaces, $f : X \to Y$ a Borel function. Is f bounded?

Theorem (D. H. Fremlin, R. W. Hansell, H. J. K. Junnila, 1983)

Assume CH, the answer is positive; assume MA(ω_1), the answer is positive too.

So far the Stone's problem is still open.

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References

- K. Ciesielski, *Set Theory for the Working Mathematician*, Cambridge University Press, 1997.
- D. H. Fremlin, R. W. Hansell, H. J. K. Junnila, *Borel functions of bounded class*, Trans. Amer. Math. Soc. 277, 1983.
- K. Gu, *Borel Functions and Separability of Metric Spaces*, Archiv der Mathematik, https://doi.org/10.1007/s00013-018-1169-x, 2018.
- A. S. Kechris, D. A. Martin, *Infinite games and effective descriptive set theory*, in: Analytic Sets (Edited by C. A. Rogers), Academic Press, London, 1980, pp. 403-470.
- D. A. Martin, R. Solovay, *Internal Cohen extensions*, Ann. Math. Logic 2, 1970, pp. 143–178.

References

- A. W. Miller, *Descriptive Set Theory and Forcing: How to prove theorems about Borel sets the hard way*, Lecture notes in logic, No. 4, Springer, Berlin, Heidelberg, New York, etc., 1995.
- A. W. Miller, *Special Subset of the Real Line*, in: Handbook of Set-theoretic Topology, K. Kunen, J. E. Vaughan (eds). Elsevier Science Publishers B.V., 1984, pp. 201–233.
- S. M. Srivastava, *A Course on Borel Sets*, Springer-Verlag, New York, 1998.
- A. H. Stone, *Some problems of measurability*, Lecture Notes in Math., vol. 375, Springer-Verlag, Berlin and New York, 1974.

The end

Thank you!