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SECOND-ORDER ABSTRACTION BEFORE AND AFTER RUSSELL'S PARADOX

Abstract In this essay, I analyze several aspects of Frege's paradigms of second-order abstraction: Axiom V and Hume's Principle. The issues dealt with include self-evidence and epistemic (non-)triviality with particular emphasis on Axiom V, Frege's attitude towards Axiom V before and after Russell's discovery of the contradiction, as well as the possible role and the status of Hume's Principle in the face of Russell's paradox. In the central part (in sections 4, 5, 6), I pursue a threefold aim: (a) to shed new light on the connection between Frege's way of introducing the primitive function-names of his formal language and the requisite self-evidence of his axioms in whose expression such a function-name occurs; (b) to analyze the semantic nature of the linguistic expression of Axiom V, and (c) to examine the conflict between the requirements of self-evidence and real epistemic value or genuine knowledge arising inevitably and invariably from Fregean abstraction principles, if they are singled out as axioms of a theory *T*. In the final section, I make critical remarks on Frege's reactions to Russell's paradox in the period 1902-1906.

1. Introduction

When Frege received Russell's famous letter of 16th June 1902, he quickly realized that his logicist project was in serious jeopardy. In particular, he felt that the paradox threw his answer to the fundamental epistemological question "How do we grasp logical objects, in particular the numbers?" into disarray. Frege's answer to this question after 1890 was: We grasp them as value-ranges of functions. More specifically, we grasp logical objects by carrying out the step of logical abstraction from right to left in Axiom V, that is, by transforming the generality of an equality of function-values into a value-range identity. Frege almost certainly knew that Axiom V could provide the appropriate epistemic access to value-ranges only if he was able to solve a burning problem arising from a semantic stipulation in *Grundgesetze*, §3. This stipulation, which was designed to govern value-ranges via their identity conditions and was later enshrined in the formal version of Axiom V, failed to fix completely the references of value-range names.[1] Yet justifying the use of canonical value-range names[2] in the formal

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[1] Perhaps even shortly after having received the bad news from Russell, Frege already had a glimmer of a suspicion that the fate of his logicism was sealed once and for all. However, in the final section of this essay we shall see that at least until 1906 Frege apparently struggled against the insight that his logicist enterprise was inevitably bound to fail.

[2] We may regard any term that results from the insertion of a monadic first-level function-name into the argument-place of the second-level function-name "$\grave{\varepsilon}\,\varphi(\varepsilon)$" (= the name of the value-range function or the value-range operator) as a *canonical value-range name*. Similarly,

language by endowing each of them with a unique reference (§§3, 10-12), and by subsequently proving this (§31), was imperative in pursuit of the logicist project.[3]

Needless to say, Russell's paradox threw into disorder not only Frege's logical construction of cardinal arithmetic, but also his theory of real numbers which he apparently had begun to work out with much confidence in the second volume of *Grundgesetze*, inspired by a clear-cut plan to bring the logicist project to a happy ending.[4] Overshadowed by Russell's paradox, this theory remained a fragment.

Facing the paradox, Frege was convinced that no scientific foundation of arithmetic would be feasible without allowing at least conditionally the transition from a concept to its extension; and he seemingly identified a scientific foundation of arithmetic with a logical one. Frege sought a way out of the quandary by modifying the (in)famous Axiom V[5]:

$$(\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\forall x(f(x) = g(x))).$$

The function-letters "*f*" and "*g*" are used here to indicate one-place functions of first level (cf. *Frege 1893*, §19); they are not variables for monadic first-level functions (cf. *Frege 1893*, §§19-20). By contrast, in the universally quantified sentence "$\forall f \forall g(\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\forall x(f(x) = g(x)))$", "*f*" and "*g*" are variables for monadic first-level functions. In his concept-script, Frege would have used German letters for the function variables. Note that nowhere in his work does the formal expression of Axiom V appear as a universally quantified (concept-script) sentence. It is always presented as an equation of the form "$a = b$", where "a" itself is

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we may call equations in which the terms flanking "=" are both canonical value-range names *canonical value-range equations*.

[3] I assume that the proof of referentiality was intended to secure also a sense for every well-formed name of the formal language: "Thus it is shown that our eight primitive names have a reference and thereby that the same holds for all names correctly formed from them. However, not only a reference but also a sense belongs to all names correctly formed from our signs" (*Frege 1893*, p. 50). Although Frege does not expressly state that it is thereby shown that all names correctly formed from the primitive names have a sense, it seems rather likely that he construes the proof of referentiality at the same time as a demonstration that all well-formed names are endowed with a sense.

[4] I say this with the proviso that Frege had perhaps intended to go even beyond the logical construction of real analysis and provide the logical foundations for the arithmetic of complex numbers as well.

[5] See the Afterword to *Frege 1903*. Strictly speaking, the formula in the next line is not Axiom V itself — which, according to Frege, is a true thought that thanks to its supposed self-evidence neither needs proof nor is capable of proof in the formal system of *Grundgesetze* — but the (formal) linguistic expression of Axiom V. For the sake of simplicity, I shall not always distinguish between the axiom and its linguistic expression. For example, I shall frequently use the phrase "the two sides of Basic Law V" and its kin. It goes without saying that in such a case I intend to refer to the linguistic expression of Axiom V. As far as the supposed self-evidence of Axiom V is concerned, see the critical discussion in section 2.

an equation of this form while "*b*" is what Frege calls the generality of an equation or of an equality (between function-values).[6] Yet it is clear that by appealing to the assumed unrestricted generality of Axiom V qua logical axiom he could have formulated it equally well as a universally quantified sentence.

Basic Law V states identity conditions for value-ranges of monadic first-level functions *f* and *g*: the value-range of *f* is identical with the value-range of *g* if and only if *f* and *g* are coextensive. In his logic after 1891, Frege notoriously construes a concept quite generally — regardless of whether it is of first, second or third level — as a function of a special type, namely as a one-place function whose value for every admissible argument is either the True or the False. Thus, Basic Law V also states identity conditions for extensions of (first-level) concepts.

Frege's intended solution to Russell's paradox was apparently to save as much as possible of the guiding idea underlying Axiom V. He divided Axiom V into Va, the right-to-left-half and its converse Vb, the left-to-right-half (cf. *Frege 1893*, p. 69). It was Vb that gave rise to the paradox, while Va can claim to be regarded as a logical truth.[7] Frege probably thought that the paradox did not affect standard axiomatic second-order logic, which together with first-order logic, constituted the ground floor of his overall logical theory. In fact, I do not know of any remark in his writings and scientific correspondence where he suggests that the inconsistency of Axiom V, when adjoined to second-order logic, casts a gloom over other second-order abstraction principles such as Hume's Principle "$N_xF(x) = N_xG(x) \leftrightarrow Eq_x(F(x),G(x))$"[8] — the exact structural analogue of Basic Law V — or even undermined

[6] Cf. *Frege 1967*, p. 130; *Frege 1976*, p. 132; *Frege 1893*, §20; *Frege 1903*, §§146-147. In the concept-script notation, Basic Law V is: $(\grave{\varepsilon}f(\varepsilon) = \grave{\alpha}g(\alpha)) = (@a@\, f(a) = g(a))$.

[7] Boolos (1997) states three reasons for justifying this claim: "(a) it is valid under standard semantics, thanks to the axiom of extensionality; (b) if the *F*s are the *G*s, as the antecedent asserts, then whatever 'extension' may mean, the extension of the *F*s is the extension of the *G*s; and (c) if the antecedent holds, then the concepts *F* and *G* bear a relation to each other that Frege called the analogue of identity" (p. 252). In the Afterword to *Grundgesetze* (*Frege 1903*, p. 257) Frege writes: "With (Vb), (V) itself collapses, but not (Va). There is no obstacle to the transformation of the generality of an equality into a value-range equality; only the converse transformation is shown to be not always allowed."

[8] In words: The number that belongs to the concept *F* is equal to the number that belongs to the concept *G* if and only if *F* and *G* are equinumerous. Frege defines the relation of equinumerosity in second-order logic in terms of one-to-one correlation (cf. *Frege 1884*, §72). Note that the above formulation of Hume's Principle is a schematic one; here its two sides are (closed) sentences, that is, "*F*" and "*G*" are schematic letters for monadic first-level predicates, not variables for first-level concepts. By contrast, in "$\forall F \forall G(N_xF(x) = N_xG(x)$

their trustworthiness. I shall discuss this topic to some extent in section 3 by drawing attention to observations on abstraction principles that Frege made in a letter to Russell of 28th July 1902 and which have largely been misinterpreted in the relevant literature.

Frege never drew any distinction between first-order and second-order logic in the sense that the latter does not enjoy the certainty and security that are characteristic of the former. The fact that is usually adduced to explain this difference, namely that second-order logic requires stronger conceptual and ontological assumptions than first-order logic, was never discussed by him.[9] To all appearances, Frege considered first-order and second-order logic to be on a par in the sense that both form the primitive or fundamental parts of logic and as such can lay claim to being unassailable. It was only the theory of value-ranges that was likely to arouse his suspicion even before he was confronted with Russell's paradox.[10] A remark at the beginning of the long Foreword to *Frege 1893* as well as a related one in the Afterword to

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$\leftrightarrow Eq_x(F(x),G(x)))$" "$F$" and "$G$" are variables for first-level concepts; here we have the universal closure of the open sentence "$N_xF(x) = N_xG(x) \leftrightarrow Eq_x(F(x),G(x))$".

[9] For some logicians, another reason for regarding second-order logic with suspicion might be its non-axiomatizability. There is of course a striking difference between first-order and second-order languages as regards their expressive power. The far-reaching expressive resources of the latter are obviously a strong plus vis-à-vis the former. Moreover, unlike first-order theories, which cannot describe a unique model up to isomorphism, unless that model is finite, theories that are framed in a second-order language can be categorical. (Incidentally, two model-theoretic theorems deal with the weaker notion of κ–categoricity for a cardinal κ. A theory T is called κ–categorical if any two models of T that are of cardinality κ are isomorphic.) In a fairly recent essay, Otávio Bueno argues convincingly that the lack of completeness of second-order logic — if the issue of completeness is put in the right perspective — is after all outweighed by the categoricity, expressive richness and manageability of this logic (see *Bueno 2010*, especially section 3, pp. 368 ff.).

[10] Among Frege scholars, there is no unanimous assessment of the (primary) cause of the contradiction in Frege's logical system (1893/1903). I, for one, share Boolos's opinion that the culprit for the breakdown of this system is what Frege took it to be, namely Basic Law V. Boolos (1993) argues convincingly that we should not put the blame for Frege's error on the stipulations he made regarding the truth-conditions of sentences beginning with second-order quantifiers, but rather on those concerning value-range equations. By contrast, Dummett (1994) has argued that the fatal flaw in Frege's system is primarily due to Frege's careless treatment of the second-order quantifier in his attempted proof of referentiality in *Frege 1893*, §31. (In my view, this metatheoretical proof, which proceeds by induction on the complexity of expressions, founders irredeemably on circularity; see *Schirn 2017*, *2017b*; see also *Heck 1997* and *Linnebo 2004*). I do not think that Dummett's assessment carries conviction. Again, I agree with Boolos when he argues against Dummett that only if the first-order fragment of Frege's system had been strong enough to yield arithmetic or an interesting portion of it, would it be tempting to trace the inconsistency back to the presence of the second-order quantifier. On the question of "how the serpent of inconsistency entered Frege's paradise" see also *Wright 2017*.

Frege 1903 are testimony to Frege's belief that the axiom governing value-ranges was not only the pivot of his logicism but also its potential Achilles' heel: "At any rate, with it the place is marked where the decision must be made" (*Frege 1893*, p. VII).

So much for the preliminaries. I shall now proceed as follows. In section 2, I make some expository and critical comments on Frege's wavering attitude towards Axiom V. Section 3 is again devoted to second-order abstraction — Hume's Principle and Axiom V — and the status it had for Frege before and after Russell's discovery of the paradox. In sections 4, 5 and 6, which form the core of this essay, I pursue a threefold aim: (a) to shed new light on the connection between Frege's way of introducing the primitive function-names of his logical system and the requisite self-evidence of his axioms in whose linguistic expressions such a function-name or more than one occur; (b) to examine the question of whether the two sides of Basic Law V are supposed to express the same thought or different thoughts and to assess the consequences that Frege has to face in each case; (c) to analyze the conflict between the requirements of self-evidence and real epistemic value or genuine knowledge arising inevitably and invariably from Fregean abstraction principles, if they are singled out as axioms of a theory *T*. In the final section, I make a number of critical remarks on Frege's reactions to Russell's paradox in the period 1902-1906. Section 7 is more or less closely related to sections 1, 2 and 3.

2. The foundational project: initial doubts and alleged irrefutability

In the formal theory of *Grundgesetze*, appeal to value-ranges based on Axiom V was needed for framing the explicit definition of the cardinality operator in purely logical terms and thus for paving the way, in accordance with Frege's logicist credo, for the derivability of Hume's Principle from a definition satisfying this constraint.[11] However, as far as the indispensability of Axiom V and, hence, of value-ranges in his foundational programme is concerned, this is not yet the entire truth. Reference to logical objects of this prototype was also required for the envisaged definition of the real numbers as ratios of quantities, as Relations of Relations, and

[11] In *Frege 1884*, §68, Frege defines the number that belongs to the concept *F* as the extension of the second-level concept *equinumerous with the concept F*: $N_xF(x) := Ext\varphi\ (Eq_x(\varphi(x),F(x)))$. "*Ext*" is here an abbreviation for the (second-level) operator "the extension of ...". In *Frege 1893*, the cardinality operator refers to a monadic *first-level* function, but the new definition of it (cf. §40) is modelled upon the pattern of the old one, its famous predecessor in *Frege 1884*. Concerning Frege's explicit definition of the cardinality operator, see the discussion in *Schirn 1983*, *1988*, *1989*, *1996a*, *2009*.

possibly for the projected introduction of the complex numbers as well.[12] Yet Axiom V plays only a rather modest formal role in the execution of that project. In fact, it no longer plays any formal role at all when Frege comes to prove the basic laws of cardinal arithmetic and analysis. His frequent "representational" usage of value-ranges — first-level functions which appear as arguments of second-level functions are represented by their value-ranges, "though of course not in such a way that they give up their places to them, for that is impossible" (*Frege 1893*, §34) — could in principle be dispensed with without any substantial loss for the logical construction of number theory and real analysis. The move of stepping down from level two to level one regarding functions is a matter of pragmatic choice, guided by the aim of achieving logical flexibility and concision.[13]

In the Afterword to *Frege 1903*, Frege confesses that he had never concealed from himself that Axiom V is not as evident (*einleuchtend*) as his other axioms and as must properly be demanded of a (basic) law of logic.[14] And he adds that he had pointed out this weakness in

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[12] The term "Relation" is Frege's shorthand expression for "*Umfang einer Beziehung*" ("extension of a relation"). Thus, in his logic *Relationen* (in English, I use "Relations" with a capital "R") are value-ranges of two-place (first-level) functions whose value, for every pair of admissible arguments (objects), is either the True or the False. Frege saw no need to introduce a special axiom governing value-ranges of dyadic (first-level) functions; he calls these *value-ranges*. Heck (*Heck 1997*, pp. 283 f.) corrrectly explains why this is so. Note in this context that the terms for value-ranges can be formed by means of the notation available for the designation of "simple" value-ranges. Regarding Frege's theory of real numbers, see *von Kutschera 1966*, *Simons 1987*, *Dummett 1991*, *Schirn 2013*, *2014*, *Synder and Shapiro 2017*.

[13] Thanks to the level-reduction regarding functions, which Frege explains in *Frege 1893*, §34, he need not introduce value-ranges of second-level functions, let alone value-ranges of third-level functions into his logical system (cf. *Schirn 2016a*). Two of the eight primitive functions of his formal system are indeed of third level. It is not clear to me why he introduces the eighth function @f@$\mu_{\beta\gamma}(f(\beta, \gamma))$ at all, since he says that he is not going to use it in his system. The seventh function is the second-order quantifier @f@$\mu_{\beta}(f(\beta))$.

[14] Frege appeals to (self-)evidence in his three principal works, in *Frege 1879, Frege 1884* and *Frege 1893/1903*. This notion also plays a certain role in several of his other writings and in his correspondence. When he deals with axioms, he usually employs the words "*einleuchtend*" or "*leuchtet ein*" and combines them occasionally with the word "*unmittelbar*" ("immediate") (see *Frege 1879*, §14; *Frege 1884*, §§5, 90; *Frege 1903*, p, 253; *Frege 1967*, p. 393; *Frege 1969*, pp. 198, 227; *Frege 1976*, p. 89). I render "*einleuchtend*" and "*leuchtet ein*" mostly as "self-evident", occasionally simply as "evident", "*unmittelbar einleuchtend*" or "*leuchtet unmittelbar ein*" as "immediately evident", and the noun phrase "*Einleuchten*" as "(self)-evidence". Regarding explicit definitions that have been turned into declarative sentences Frege usually employs the word "*selbstverständlich*". For the sake of distinguishing it from "*einleuchtend*", I render it as "obvious" (cf. *Frege 1967*, pp. 263, 290; *Frege 1969*, pp. 167, 225; *Frege 1976*, p. 62). In one place (*Frege 1893*, §50), Frege also applies the word "*selbstverständlich*" to a theorem, namely to "$a = a$", which he proves nonetheless. He

the Foreword to *Frege 1893* (see p. VII). If we take him at his word, then the last statement defies credibility. In the Foreword, Frege does not draw attention to any specific shortcoming in Axiom V, let alone to its lack of the required degree of self-evidence. At the very most we can say that in the Foreword he expresses a possible or an unspecified concern about Axiom V. However, Frege does not expressly say that he himself surmises that something might go awry with Axiom V, which could hardly be reconciled with the unshakable confidence he seems to have had in his logicist project, if we give credence to what he says in this respect at the end of the Foreword (p. XXVI). He is rather envisioning a potential controversy about Basic Law V, not necessarily provoked by himself, when the grounds on which each individual theorem rests — the basic laws, the definitions and the rules of inference — are in the focus of attention.[15] Unfortunately, Frege does not explain why he thinks that a dispute might be roused. Is he imagining a possible debate regarding the assumed purely logical nature of Basic Law V, as his seemingly reassuring assertion "I take it to be purely logical" in the Foreword and elsewhere (*Frege 1967,* p. 130; *Frege 1903*, §146; *Frege 1969*, p. 198) might suggest — if we interpret it as a sign of insecurity, rather than an expression of certainty — or concerning its requisite (degree of) self-evidence, as his remark in the Afterword strongly suggests? Or does Frege think that in the case of Basic Law V these two issues are inextricably intertwined? We do not know. To be sure, unless we consider propositions that we characterize as *primitive* truths of logic by invoking Frege's criteria for truths of this kind, a proposition need not be self-evident in order to be classified as a logical truth. As a matter of fact, the vast majority of the theorems that he proves in *Grundgesetze* are not self-evident — hence, the necessity to prove them, to justify them deductively and to

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employs "*selbstverständlich*" for the most part, if not always, in the sense of "goes without saying", "is epistemically trivial", "is tautological" (cf. *Frege 1884*, §67; *Frege 1893*, §50; *Frege 1967*, pp. 263, 290; *Frege 1969*, p. 225; *Frege 1976*, pp. 62, 128, 234 f.). A truth which is obvious in this sense is also (self-)evident, but the converse does not hold generally, according to Frege. In his letter to Husserl of 9.12.1906, Frege uses the phrase "*logisch evidenter Sinnbestandteil*" ("logically evident sense-component"). This is the only significant place that I know where he uses the German word "*evident*" at all. It is clear that for Frege the self-evidence of a truth cannot serve as a general criterion of analyticity. On the one hand, he grants that there are non-evident sentences which are analytic truths, such as, for example, the equation "125664 + 37863 = 163527", provided that the logicist programme has been successfully carried out for cardinal arithmetic. On the other hand, Frege acknowledges the existence of self-evident, but non-analytic truths, such as the axioms of Euclidean geometry. For a true statement "$a = b$" to be analytic in Frege's sense, the identity of the sense(s) of "a" and "b" is a sufficient condition, but it is not a necessary one.

[15] Frege formulates as follows: "Ein Streit kann *hierbei* [my emphasis], soviel ich sehe, nur um mein Grundgesetz der Werthverläufe (V) entbrennen ..." The word "*hierbei*" relates to a consideration of the grounds on which each individual theorem rests.

establish in this way their purely logical character. In sections 4, 5 and 6, I shall discuss various aspects of the notion of self-evidence with regard to Basic Law V.

On the face of it, the sentence "Jedenfalls ist *hiermit* [my emphasis] die Stelle bezeichnet, wo die Entscheidung fallen muss", which follows immediately after "Ich halte es für rein logisch" ("I take it to be purely logical") is not free from ambiguity. I favour the translation "At any rate, with it the place is marked where the decision must be made" (cf. section 1), because I assume that by using the word "*hiermit*" Frege intends to refer to Basic Law V as such and not exclusively to its supposed purely logical character. In particular, I do not think that the quoted sentence beginning with "*Jedenfalls*" furnishes conclusive evidence that Frege had doubts about the logical nature of Basic Law V. It would indeed be strange if he said that he takes Basic Law V to be purely logical and in the same breath added that he is nevertheless worrying about its status as a (primitive) truth of logic. Even if he intended to convey that it is the logical nature of Basic Law V, rather than its requisite self-evidence, that marks the spot where the decision — about the viability of his logicist project — must be made, this would not imply that he himself had misgivings about the claimed purely logical nature of Basic Law V. To be sure, Frege does not speak of a *weak* spot where there has to be a decision, but he probably knew as much as this: If one of his fellow logicians had turned up to rouse a dispute about Basic Law V, say, shortly after the publication of the first volume of *Grundgesetze*, the point at issue might well have been its status as a truth of pure logic. As I shall argue below, we cannot rule out that Frege was *secretly* concerned about the status of Basic Law V as a logical truth without having the courage to spell this out. However, in this context we must also bear in mind that in hindsight he confesses only that he had never concealed from himself that Basic Law V lacks the requisite degree of self-evidence. And recall that according to Frege self-evidence is a key prerequisite only for a primitive truth, in particular, for one that has been laid down as an axiom of a logical or a geometric theory. In this respect, there is no difference between logical and geometric axioms, but of course they differ regarding their degree of generality and the sources of knowledge to which they belong.

I am keen to say a bit more at this stage about the assumed purely logical nature of Basic Law V; I shall resume this topic from different points of view in later sections. To begin with, contrary to what some Frege scholars suggest, *nowhere* in his writings and correspondence before 1902, including the Foreword to *Grundgesetze*, does Frege raise explicit doubts whether Axiom V is purely logical, in particular, whether it meets the requirement of unbounded generality and universal applicability. If before Russell's discovery of the paradox he was secretly in doubt about the status of Basic Law V as a logical truth, he might have thought, for example, that this law does not determine our reasoning in that fundamental and comprehensive way which is characteristic of the law of excluded middle, the law of identity $a = a$ or any other basic law of classical two-valued logic. A closely related source of concern regarding the logical status of Basic Law V might have been the suspicion that the value-range function is not as proper to logic as, for example, negation, the conditional, identity and the quantifiers. The latter notions are not only more directly, but also more generally involved in our rational thinking and deductive reasoning than the former notion. In 'Über die Grundlagen der Geometrie' II (1906), Frege underscores that logic has its own concepts and relations such as negation, identity, subsumption, etc. for which it allows no replacement. And he takes this to be an unmistakable mark that the relation of logic towards what is proper to it is not at all formal. In the light of these remarks, one might raise the question of whether in *Grundgesetze* Frege really did consider the value-range function to belong intrinsically and irreplaceably to logic, that is, to be on a par with negation, identity, etc. Furthermore, it is conceivable that he was worried about the fact that, compared with his axioms for first- and second-order logic (axioms I — IV), Axiom V has a lower degree of generality, since it was designed to hold only in the domain of value-ranges. And how about the fact that value-ranges, in contrast to functions, concepts and relations, have only a derivative nature? Did this perhaps arouse suspicion that Axiom V was not purely logical? Last but not least, Frege might have been concerned about the fact that Axiom V involves a massive ontological commitment which could affect its status as a primitive truth of logic as well. Admittedly, due to the fact that Frege keeps his cards close to his chest, all this is highly speculative, but it is not totally ungrounded. If he had one or more of these concerns in mind when he wrote the Foreword to *Grundgesetze*, he should have tried to get this off his chest instead of proclaiming with apparent bravado: "I take it [Basic Law V] to be purely logical."[16]

Tyler Burge (1998, p. 348) observes that Frege probably did worry about the logical status of Axiom V independently of worrying about its truth. But he believes that Frege was uneasy about its truth as well, which in the light of Frege's remarks on the unassailableness of his logicist project at the end of the Foreword to *Grundgesetze* I take to be unlikely, at least if we assume that these remarks result in fact from an imperturbable conviction. (In a moment, I shall, however, relativize this assumption.) In an earlier article (*Burge 1884*, p. 24), Burge contends that Frege's willingness to replace "the extension of the concept *F*" by "the concept *F*" in his explicit definition of the cardinality operator in *Grundlagen* reflects Frege's

[16] Why? No answer is given by Frege either in *Grundgesetze* or in any other of his writings.

"struggle to justify Law (V) as a logical law". Again, this is sheer phantasy. Frege does not yet fomulate Basic Law V in *Grundlagen*, although he arguably comes close to formulating its third-order cousin for *extensions* of second-level *concepts*: $Extf(M_\beta(f(\beta))) = Extf(N_\beta(f(\beta))) \leftrightarrow \forall f(M_\beta(f(\beta)) \leftrightarrow N_\beta(f(\beta)))$ (see *Grundlagen*, §73 and the comments in *Schirn 2016*). And as far as *Grundgesetze* is concerned, there is not a trace of evidence that Frege *struggled* to justify the logical status of Axiom V. Why *must* the general equivalence of the coextensiveness of two monadic first-level functions and the identity of their value-ranges be regarded as a logical law as Frege asserts (*Frege 1967*, p. 130; *Frege 1893*, §9)?[17] He emphasizes (again in §9) that use has always been made of this equivalence in its less general form, even if tacitly, whenever extensions of concepts were at issue (cf. *Frege 1903*, §147). Yet this is hardly a justification of the required kind, and he must have been aware of this.

With respect to the formal expression of Axiom V, we may distinguish between its "external" structure or form (= an identity statement of the form "$a = b$") and its "internal" structure or form: "a" is of the form "$\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)$" and "$b$" is of the form "$\forall x(f(x) = g(x))$". Obviously, if Frege were to appeal only to its external form, he could not say that Basic Law V is true by virtue of its form and, hence, can be acknowledged as a truth of logic. However, by invoking its internal structure, he might have considered Basic Law V to be true by virtue of its form, since it was designed and actually taken to be true for every pair of sharply defined monadic first-level functions and their value-ranges.

In one place, a few years after Russell's discovery of the paradox, Frege concedes by wisdom of hindsight that he has committed the error of giving up too lightly his initial doubts about Axiom V by trusting in the fact that in logic extensions of concepts have been talked about for a long time.[18] I presume that here again (as in the Afterword) he appeals to his worry about the lacking (degree of) self-evidence of Axiom V, although he does so only implicitly. Finally, in his fragment 'Über Schoenflies: Die logischen Paradoxien der Mengenlehre' (*Frege 1969*, p. 198), Frege almost exactly repeats his earlier verdict about Axiom V in the Afterword by saying that it is not as evident as one would wish for a law of logic. Thus, there are only two places in his entire work where he raises the issue of self-

[17] In *Frege 1903*, §146, Frege observes (I slightly paraphrase): That we have the right to acknowledge what is common to two coextensive (monadic first-level) functions, namely their value-range, and that, accordingly, we can transform the generality of a function-value equality into a value-range identity, *must* be regarded as a basic law of logic.

[18] *Frege 1976*, p. 121. Frege makes this concession in a comment on an account of his own work written by P. E. B. Jourdain.

evidence in relation to Axiom V in explicit form, and he does so only after having heard from Russell about the contradiction in the system of *Grundgesetze*.

Let us charitably assume that in composing the Foreword to *Grundgesetze* Frege was indeed aware that Axiom V lacked the requisite (degree of) self-evidence, despite the fact that in the Foreword he passes over this concern in conspicuous silence. Consider now the claim that he makes at the very end of the Foreword about the alleged irrefutability of his project. There (*Frege 1893*, p. XXVI) he claims — this time with exaggerated bravado — that no one will succeed in refuting his foundational project, be it by showing that on the basis of different fundamental convictions a better, more endurable system than that of *Grundgesetze* could be worked out, or by demonstrating that the basic principles of *Grundgesetze* lead to manifestly false conclusions. It springs to mind that Frege's flamboyant remark on the unsinkable ship of his logicism stands in striking contrast to the assumption I made above. In the light of what he says in the Afterword to *Grundgesetze* about the self-evidence of Axiom V with an eye to the Foreword, the claim may even sound like whistling in the dark.

So, Frege considers here two possible ways of acknowledging a refutation of his logicist project. However, it remains unclear what he means exactly by erecting a possibly better, more enduring edifice on different fundamental convictions and why he should recognize it as a *refutation* of his foundational work, if he had to face it. By "different fundamental convictions" Frege probably means "different axioms, different rules of inference, different definitions". We know that in *Grundgesetze* he relies by hook or by crook on value-ranges. Imagine now, for the sake of argument, that young Russell in his early twenties by and large endorsed the logical theory of *Grundgesetze*, but inventive and ingenious as he was, suggested to Frege that he should replace Axiom V with an axiom that likewise governed value-ranges but, unlike Axiom V, fully met Frege's requirement of self-evidence. Suppose that in the early 1890s Frege was indeed seriously worried about the lacking self-evidence of Axiom V, and suppose further that he would have accepted the suggestion with a sigh of relief. Would he have regarded the replacement as a refutation of his project? Probably not. Lack of the required degree of self-evidence of Axiom V did not necessarily undermine logicism, nor was it automatically a threat to the truth of Axiom V. By contrast, if someone had succeeded in proving that the basic principles of *Grundgesetze* lead to manifestly false conclusions, this obviously would have been a refutation of Frege's project. Similarly, suppose that one of his contemporaries had argued conclusively and incontestably that Basic Law V is not a truth of pure logic. Again, Frege would have been compelled to acknowledge this as a refutation of his logicism, though he may have found it less devastating than Russell's discovery in 1902.

I conclude this section with remarks on the truth-values and value-ranges, the only objects whose existence was required by the axioms of *Grundgesetze*. To all appearances, Frege regarded value-ranges as a prototype of logical objects, but assigned the status of being a *primitive* logical object to the two truth-values, although not in quite explicit form. He felt entitled to assume that everybody, even the sceptic about truth and falsity, is already familiar with the True and the False in his or her ordinary practice of judging and asserting. This may partly explain why the truth-values had a privileged and safe place in his mature logic after 1890. I say this despite the fact that in *Grundgesetze*, §10 Frege identified the True and the False with special extensions of concepts, indeed with their unit classes. As far as the more general transsortal identification made in *Grundgesetze* is concerned, namely the (projected) identification of all numbers, not only of the cardinals, with value-ranges, it was intended to secure and justify the assumed purely logical status of the numbers and at the same time to make possible a general answer to the question "How do we to grasp the numbers?" as opposed to individual and separate answers for each class of numbers.[19] By contrast, the objective of the identification of the truth-values with value-ranges was not to underpin, let alone establish the logical status of the former. This identification was only intended to remove in a first crucial step the referential indeterminacy of canonical value-range names deriving from the semantic stipulation in *Frege 1893,* §3. There was no need or possibility for Frege to enhance the logical nature of the truth-values by way of identifying them with extensions of concepts.

Unlike the value-ranges that derive from something more fundamental in logic, namely functions, Frege regarded the truth-values as being on a par with the primitive functions out of which he intended to develop the whole wealth of objects and functions dealt with in mathematics, as from a seed. Moreover, in contrast to the value-ranges, the truth-values did not give rise to any indeterminacy or underdetermination. And, unlike the status of the value-ranges, the eminent status of the truth-values could survive in the face of the set-theoretic paradox, provided that Frege was prepared to cancel their identification with their unit classes.[20] His comment (probably written in 1910) on an account of his own work written by

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[19] This is one of several reasons why in the aftermath of Russell's paradox Frege could not have "replaced" Basic Law V with Hume's Principle which governs only cardinal numbers; see in this respect the end of section 3.

[20] In the Afterword to *Frege 1903*, Frege still thought that these identifications "suffer no alteration under the new conception of the extension of a concept" that he had suggested in the Afterword. However, this proved to be an illusion.

P. E. B. Jourdain is illuminating in this respect and also lends support to what I have said above. He writes (*Frege 1976*, p. 121)[21]:

> The laws of number were supposed to be developed purely logically. Yet the numbers are objects, and in logic we initially have only two objects: the two truth-values. Thus the quite obvious thing was to generate objects from concepts, namely extensions of concepts or classes…The difficulties that are connected with the use of classes disappear if one deals only with objects [here Frege is probably alluding exclusively to the two truth-values], concepts, and relations, which is possible in the fundamental part of logic. Of course, the class is something derived whereas in the concept — as I understand the word — we have something primitive. Accordingly, the laws of classes are less primitive than the laws of concepts, and it is inappropriate to base logic on the laws of classes. The primitive laws of logic must not contain anything that is derived.

Especially in an earlier version of his comment, Frege underscores that when he embarked on writing *Grundgesetze* he decided to permit the transition from concepts to their extensions, but not for internal, intrinsically logical reasons. In other words, he did not intend to extend and strengthen logic just for its own sake by adjoining to it a theory of classes and then, thanks to a lucky coincidence, derive from this the additional benefit of grounding arithmetic on logic alone. As I suggested earlier, Frege's principal motive for developing his theory of value-ranges as a generalized theory of classes was expressly his conviction that his treatment of the numbers required a means of introducing objects in a purely logical fashion and in this way to guarantee our non-intuitive cognitive access to them.[22] The aim of achieving some simplifications in the logical construction of arithmetic by using value-ranges was only of secondary importance for him. In summary then, considered by itself, logic does not stand in need of dealing with classes. It is worth noting that in the course of critically assessing his logicist project in the light of Russell's paradox Frege never declared, nor even insinuated, that he went astray in associating the most salient feature of logic, which is utmost generality, with the nature of arithmetic. It was precisely the insight that logic and arithmetic share the

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[21] All translations from Frege's works are my own.

[22] In *Schirn 2006a*, I have argued at length against *Ruffino 2003* that Frege does not introduce extensions of concepts into his logical theory because he believes that they are badly needed to account for a characteristic defect in the grammatical structure of natural language and, therefore, quite independently of his logicist manifesto. It is likewise obvious that Frege is not obliged to "buy" extensions for his logical calculus, whatever the cost, in order to be able to make "indirect" statements about concepts. From what I have said above it should also be fairly clear that in *Grundgesetze*, in contrast to *Grundlagen*, Frege does not introduce extensions of concepts with the purpose of solving a kind of Julius Caesar problem with respect to the introduction of cardinal numbers. Such a problem simply does not arise in *Grundgesetze* (see section 3).

feature of unrestricted generality that generated the idea that logic alone is at the root of arithmetic.

So much at this stage for some of the cornerstones of Frege's foundational project with special emphasis on his wavering attitude towards Axiom V. I shall now take a closer look at some aspects mainly of Hume's Principle. In doing so, I shall make further comments on Axiom V, the central topic of this essay. As before, I shall consider both the pre- and the post-paradox period.

3. Frege's paradigms of second-order abstraction: Hume's Principle and Axiom V

Unlike the first-order abstraction principles, which in his logicist project Frege uses only for the sake of illustration, the principles of second or higher order involve a projection from the larger domain of concepts (or functions) into the smaller domain of abstract or logical objects of a certain kind; and the latter may, of course, fall under the former, if they are of first level. It is arguably this feature of the higher-order principles that makes them fairly powerful, but at the same time susceptible to logical difficulties. I imagine that this difference between first-order and second-order abstraction did not go unnoticed by Frege, although he does not mention it.[23] In particular, I assume that he did not think, either before or after Russell's discovery of the paradox, that a properly formulated first-order abstraction principle might turn out to be inconsistent. While Frege's preferred example of an abstraction principle for the purpose of illustration in *Grundlagen* — namely the transformation of the relation of parallelism between lines into an identity between directions, and *vice versa* — does not require the existence of any more abstracta (directions) than there are lines, Hume's Principle requires the existence of $n + 1$ abstracta (cardinal numbers), given n objects of the original kind. In fact, Frege's introduction of the cardinality operator relies crucially on the assumption that the first-order variables range over an infinite domain. As far as Axiom V is concerned, he had to learn that the demand it makes on the size of the domain is not realizable. If n objects are in the domain, Axiom V requires the existence of 2^n abstracta. If

[23] Frege does not use the term "abstraction" when he is concerned with what we nowadays call *Fregean abstraction*. He probably thought that due to his rejection of what he considered to be misguided methods of abstraction (cf. *Frege 1884*, §21, §34, §45, §§49-51; *Frege 1967*, pp. 164 f., 186 ff., 214 ff., 240-261, 324 ff.) the term "abstraction" had acquired a negative meaning, at least for himself. When he comments with plenty of sarcasm on Cantor's method of obtaining the ordinal or cardinal number of a set via a single or a double act of abstraction, he says that the verb "to abstract" is a psychological expression and, as such, to be avoided in mathematics. Thus, with the exception of what *we* call Fregean abstraction, it seems that Frege regarded abstraction in its manifold guises as a thorn in the flesh.

the number of abstracta introduced via an abstraction principle exceeds the number of objects in the domain, we may call the principle *inflationary*, following a proposal made by Kit Fine (1998).[24]

Due to the lack of available evidence, we cannot rule out that around 1884 Frege had already an inkling that Hume's Principle, taken in its role as a (tentative) contextual definition of the cardinality operator, gives rise not only to the apparently intractable Julius Caesar problem (which likewise affects first-order abstraction principles), but also to one or the other additional difficulty. Naturally, I do not mean the first two of three logical doubts that Frege raises in *Grundlagen*, §§65-66 when he comes to consider the contextual definition of the direction operator and by analogy that of the cardinality operator.[25] In fact, the first two doubts are innocuous or even spurious. The third logical doubt discussed by Frege is the

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[24] Fine (1998, p. 510) proposes, by appeal to an informal concept of truth, that an abstraction principle will be true if and only if its identity criterion is non-circular and yields a non-inflationary and predominantly logical equivalence on concepts. He calls an abstraction principle *predominantly logical* if its identity criterion involves only a "small" number of objects in relation to the number of objects in the universe as a whole. Notice that the notion of being small that Fine uses here is not the usual one. A subset *C* of cardinality *c* is said to be small relative to a domain *D* of cardinality *d* if $d^c \leq d$, that is, if the number of subsets of the same cardinality as the given subset does not surpass the cardinality of the domain itself. See Fine's comparison of two model-theoretic criteria of acceptability for abstraction principles with the aforementioned informal criterion, pp. 511 ff. Philip Ebert (*Ebert 2008*) suggests that it is not entirely correct to say that Hume's Principle is inflationary in Fine's sense of this term. He points out that in order to settle this issue one must also look at certain background assumptions. Ebert claims that in an Aristotelian universe, where there are no empty concepts, Hume's Principle is not inflationary. As regards the existence of properties over which the higher-order quantifiers range, Ebert distinguishes between an Aristotelian conception and a Platonist conception. While according to the Aristotelian conception properties can be said to exist only if they are concretely instantiated, the platonist does not impose any such restriction on the existence of properties. Ebert argues that when embedded in Aristotelian logic with a restricted comprehension scheme (see p. 214), Hume's Principle is not "presumptuous"; "it will never inflate an originally finite domain to an infinite one .... It does not involve the existence of *too many objects*, *i.e.*, $\aleph_0$-many objects." (Ebert explains the criterion of presumptuousness as follows (p. 212): Assuming on purely analytic grounds the existence of a function [for example, the cardinality function governed by Hume's Principle] is *presumptuous*, if and only if its application has *further ontological commitment* on the object level.) This result seems to reinforce the idea that Hume's Principle involves the existence of infinitely many objects only when it is combined with additional (metaphysical) assumptions concerning the existence of properties.

[25] Note that Frege defines "*n* is a cardinal number" ("$CN(n)$"), "0", "1" and "∞_1" (or "$\aleph_0$") in terms of the cardinality operator "$N_x\varphi(x)$": $CN(n) := \exists\varphi(N_x\varphi(x) = n)$; $0 := N_x(x \neq x)$; $1 := N_x(x = 0)$; $\infty_1 := N_xFCN(x)$, where "*FCN*" is shorthand for "finite cardinal number".

Julius Caesar problem, and it is the only doubt that he upholds.[26] In what follows, I shall assume familiarity with the informal version of the Caesar problem in *Grundlagen*, §66 and shall only occasionally touch upon it and roughly characterize its re-emergence in a formal guise in *Grundgesetze*. In the remainder of this section, I want to discuss some special aspects regarding both Hume's Principle and Basic Law V by way of scrutinizing in the first place a passage in Frege's letter to Russell of 28th July 1902 in which Frege hints at the difficulties connected with Axiom V and abstraction in general. In a second step, I shall chiefly analyze the nature and role of Hume's Principle in Frege's logicist programme.

In a written discussion on some aspects of Fregean abstraction principles, Patricia Blanchette drew my attention to the passage that I mentioned above. She points out that she has always thought of it "as an indication that, for Frege, the damage done by the paradox is quite widespread, at least with respect to the confidence one might have in the reliability of such principles."[27] Here then is the relevant passage of Frege's letter:

> I myself was long reluctant to acknowledge value-ranges and hence classes; but I saw no other possibility of placing arithmetic on a logical foundation. But the question is: How do we grasp logical objects? And I have found no other answer to it than this: We grasp them as extensions of concepts, or more generally, as value-ranges of functions. I have always been aware that there are difficulties connected with this, and your discovery of the contradiction has added to them; but what other way is there?... We can also try to help ourselves in the following way [Man kann sich auch so zu helfen suchen] and I hinted at this in my *Grundlagen der Arithmetik*. If we have a relation $\Phi(\xi, \zeta)$ for which the following sentences hold: 1. from $\Phi(a, b)$ follows $\Phi(b, a)$; 2. from $\Phi(a, b)$ and $\Phi(b, c)$ follows $\Phi(a, c)$, then this relation can be transformed into an equality (identity), and $\Phi(a, b)$ can be replaced by saying, e.g., $\S a = \S b$. If the relation is that of geometrical similarity, then for "*a* is similar to *b*" we can say "the shape of *a* is the same as the shape of *b*". This is perhaps what you call "définition par abstraction". But the difficulties here are the same as in the transformation of a generality of an equality into a value-range equality.

Unfortunately, what Frege writes here is more intricate than meets the eye, and what is worse, it is marred by vagueness. So we must take a close look at the passage.

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[26] In my view, it would be anachronistic to assume that Frege was aware of all the key issues that have been discussed with respect to Hume's Principle during the last twenty years, including the problem that Hume's Principle, as conceived of by Frege, is irredeemably impredicative (see *Dummett 1998*, *Wright 1998*, *1998a*) and what has come to be known as the bad company objection (see in this respect the papers by different hands in *Linnebo 2009*).

[27] I quote here with her permission. In a second interesting comment, Blanchette characterizes her position in more detail. Unfortunately, I do not have the space here to discuss her remarks appropriately, and it is for this reason that I shall not quote from her second comment. I think that the difference between our views will emerge in this section.

After having raised what he regards as the fundamental problem of arithmetic — the problem of how we manage to have cognitive access to logical objects, in particular to the numbers — Frege confesses that he has always felt that Basic Law V, designed as the appropriate means of coming into epistemic contact with logical objects, gives rise to difficulties. What difficulties does he mean? If we give credence to his remarks, lack of the requisite (degree of) self-evidence of Axiom V appears to have been the only serious concern before 1902.[28] Frege apparently believed that the original Caesar or indeterminacy problem in *Grundlagen*, stemming from a contextual definition of a term-forming operator via an abstraction principle, was removed thanks to the final explicit definition of the relevant operator. When in *Frege 1893*, §10 he faces a formal version of his old Caesar problem concerning now value-ranges, he is confident that it can likewise be resolved, although in an entirely different fashion than in *Grundlagen*. The obvious reason for this change is that the value-range operator "the value-range of the function φ" (in symbols: "$\acute{\varepsilon}\varphi(\varepsilon)$"), unlike the cardinality operator, figures as one of the primitive function-names from which Frege unfolds his logical system. The strategy is now to determine for every first-level function, when introducing it, which value it obtains for value-ranges, just as for all other suitable arguments (that is, objects).[29] For the sake of brevity, I call this "the procedure of function-value determination", notably for primitive first-level functions. But the worry that Axiom V is not as evident as the other axioms of Frege's system and as he would wish for a primitive law of logic persisted and put a damper on things. Note that in Frege's view self-evidence was seemingly a safeguard against falsity, and thus he *might* have thought that lack of the requisite (degree of) self-evidence made Axiom V vulnerable to falsity.[30]

In the letter under consideration, Frege goes on to mention that Russell's definition of the cardinal number of a class *u* agrees with his own explicit definition of the cardinality operator.

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[28] If in composing the Foreword to *Frege 1893* Frege was seriously worried about the truth of Basic Law V — which I take to be unlikely (cf. section 2), but do not categorically rule out — his remark at the end of the Foreword on the alleged irrefutability of his logicist project would appear outlandish.

[29] Later I shall say a few explanatory words about the referential indeterminacy of value-range terms and Frege's strategy of removing it.

[30] Neither in *Frege 1903*, p. 253 nor in *Frege 1969*, p. 198 does Frege speak explicitly of degrees of evidence. Yet this does not mean that he did not have degrees in mind. The comparative form "not as evident as the other axioms" that he uses strongly suggests that he believed in degrees of evidence; in fact, he does not say that Axiom V is not evident at all. In section 6 of this essay, I shall appeal to degrees of evidence when I introduce Charles Parsons's notion of intrinsic plausibility and the alternative notion of logical transparency coined by myself.

He adds the proviso that classes must not be regarded as systems or aggregates or wholes consisting of parts.[31] In what follows, Frege somewhat abruptly observes that we can also try to help ourselves in the following way. Yet instead of explaining to Russell the point he wishes to make, he characterizes Frege-style abstraction in general by taking geometrical similarity as an example. *Firstly*, there was of course no need to explain to Russell the nature of equivalence relations and the transition from such a relation to an identity of abstracta. *Secondly*, it remains obscure why Frege describes the nature of abstraction without giving at least a hint of how he might find a way out of the impasse to which his definition of the cardinal number belonging to a concept (or a class) as an equivalence class of equinumerosity (or similarity) has led in the face of Russell's paradox.[32] The impasse stems from the fact that the *definiens* is framed in terms of extensions of concepts and, hence, rests on the inconsistent Axiom V. Russell may have expected that Frege's announcement "We can also try to help ourselves in the following way" is followed by a constructive proposal of how one might tackle or even solve the problem that derives from Axiom V. *Thirdly*, why does Frege not mention a second-order abstraction principle, preferably Hume's Principle? It is plain that only such a principle might be considered at all a candidate for the introduction of cardinal numbers as logical objects without recourse to classes.[33]

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[31] The observations that Frege makes both in *Frege 1884*, §§23, 28 and in his essay 'Über formale Theorien der Arithmetik' of 1885 (*Frege 1967*, pp. 104 f.) leave no doubt that around 1884 he had already banished the notion of an aggregate from logic. After 1892, he dismissed Schröder's conception of classes as collective units or aggregates (*Frege 1967*, pp. 193-210), Dedekind's conception of systems (*Frege 1893*, pp. 1-3), and also Russell's conception of classes as aggregates, systems or wholes consisting of parts (*Frege 1976*, pp. 222 f., 225). Frege did so mainly on the grounds that their systems or aggregates do not belong to logic at all: "Empty concepts [or empty extensions] are possible, empty aggregates are absurdities" (*Frege 1903*, §150).

[32] Incidentally, Russell would have been in a position similar to Frege's had he grounded his explicit definition of the cardinality operator on an abstraction principle along the lines of Frege's Axiom V, let us say, on a principle that was restricted to the coextensiveness of concepts and the identity of their extension(s).

[33] In his letter to Frege of 24th May 1903, Russell writes that he believes he has discovered that classes are entirely superfluous. He defines the relation of similarity between concepts, that is, what Frege calls the relation of equinumerosity, in terms of one-to-one correlation and the cardinal number of φ as the class of concepts similar to φ. In Russell's notation, the latter definition is: $Nc(\varphi) = \psi'(\varphi \text{ sim } \psi)$. Russell adds: "We have $\vdash: \varphi \text{ sim } \psi \,.\equiv.\, Nc(\varphi) = Nc(\psi)$. In this way we can do arithmetic without classes. And this seems to me to avoid the contradiction" (*Frege 1976*, pp. 241 f.). I presume that by saying "We have $\vdash: \varphi \text{ sim } \psi \,.\equiv.\, Nc(\varphi) = Nc(\psi)$" Russell wants to convey that the equivalence of the similarity between φ and ψ and the identity of the cardinal numbers $Nc(\varphi)$ and $Nc(\psi)$ (that is, Hume's Principle) can be derived from his definition of "$Nc(\varphi)$". Note in this entire context that in *Russell 1903*

"But the difficulties here are the same as in the transformation of a generality of an equality into a value-range equality." Frege is here lumping together the difficulties Axiom V gives rise to with those arising from the transformation of the relation of geometrical similarity into an equality of shapes (and *vice versa*). What difficulties does he have in mind and what justifies lumping them together? And what conclusion is Russell supposed to draw from Frege's quasi-elliptical statement? Regrettably, Frege refrains from putting his cards on the table. On the face of it, his remark sounds like a capitulation. It seems that any initial hope, however faint or vague, of finding a way out of the predicament by replacing Axiom V with another abstraction principle (qua definition or qua axiom?) — and this is all that Frege is insinuating — is dashed.[34]

Needless to say, the difficulties plaguing Axiom V are not exactly the same as those involved in a first-order abstraction principle, nor do they coincide with those arising from Hume's Principle. Axiom V had fallen prey to inconsistency, while Hume's Principle had not fallen into similar disrepute. If we take Frege's remark "But the difficulties here are the same…" at face value, then he seems to be advancing the thesis that the blemish of inconsistency affects abstraction principles across the board, contrary to what I take him to believe. However, it would have been out of character for him to have made such a slip. So, I assume that the apparent discrepancy is due to an infelicitous choice of phrasing. In any event, the inconsistency of Axiom V is after all the only damaging difficulty that Frege ever had to face when he dealt with logical abstraction. As to the requirement of self-evidence that he

(pp. 114 f.) Russell rejected definitions by abstraction ("définitions par abstraction"). He argued that such definitions suffer from an "absolutely fatal formal defect"; they do not show that only one object satisfies the definition.

[34] In his paper 'Julius Caesar and Basic Law V' (*Heck 2005*; cf. *Heck 2011*, chapter 5), Heck makes a number of claims that to my mind reverse the true order of things; not all of them are new (cf. *Heck 1995*). One of Heck's claims is as follows: "... not only did Frege know that he could have substituted HP for Basic Law V, he *explicitly* [my emphasis] considered doing so" (p. 163). Heck refers here to the passage in Frege's letter to Russell under discussion. I take the first half of his claim to be grossly misleading (see my argument at the end of section 3) and the second half to be undermined, if not refuted by what I said above. Another of Heck's assertions in this connection is: "In the cited letter to Russell, Frege remarks that there are certain difficulties connected with adopting HP as a primitive axiom ..." (p. 164). Following my analysis of the passage under consideration, this claim is likewise off target. Heck further contends: "It is ... the so-called Julius Caesar problem, that prevents Frege from ... adopting it [HP] as an axiom" (p. 164). Again, in the light of the available evidence, this thesis is far-fetched and probably untenable. I discuss these and related theses put forward in *Heck 2005* in my note 'Heck on Julius Caesar and Basic Law V' (unpublished). Regarding the interpretation of Frege's letter to Russell see also *MacFarlane 2002* and the critical discussion of his view.

regards as crucial in the case of Axiom V and for axioms in general, it constitutes an intrinsic difficulty neither for Hume's Principle nor for any first-order abstraction principle. Only if Frege had decided to lay down Hume's Principle as a logical *axiom* of his theory would it have been essential for him to make sure that it met the demand of self-evidence. I trust that Hume's Principle, albeit consistent, would not have fared much better in this respect than Axiom V. I shall say a little more about this in a moment.

In *Grundlagen*, Hume's Principle generated only one genuine difficulty, namely the obstinate Julius Caesar problem. The latter was apparently omnipresent in Fregean abstraction, regardless of whether an abstraction principle appeared in the guise of a (tentative) contextual definition, as did initially Hume's Principle in *Grundlagen*, or was clad in the garb of an axiom, as was the case with the transformation of the coextensiveness of two monadic first-level functions into a value-range identity, and *vice versa* in *Frege 1893*. Remember that the first two difficulties that Frege mentions in connection with his tentative contextual definitions are harmless or even spurious and, hence, totally irrelevant when in 1902 the problem "How to save the logicist project in the face of Russell's paradox?" was on the agenda.

I wish to argue that in spite of all the similarity between Frege's approaches to cardinal arithmetic in *Frege 1884* and *Frege 1893,* it would be patently false to transfer the close connection that we find in *Frege 1884* between the emergence of the Caesar problem from Hume's Principle and its attempted solution via the explicit definition of the cardinality operator to *Frege 1893*. The situation in *Frege 1893* differs significantly from that in *Frege 1884* as far as the Caesar problem and its alleged solution are concerned. Let me make one point to illustrate this. In the course of introducing and proving Hume's Principle in *Frege 1893* Frege does not come across a Caesar problem concerning cardinal numbers at all. It is only in §53 that he introduces Hume's Principle, presenting now a new version of it which is equivalent to the old one in *Grundlagen*: The cardinal number of a concept is equal to the cardinal number of a second concept, if a relation maps the first into the second, and if the converse of this relation maps the second into the first. At this stage, cardinal numbers are already defined as special value-ranges, as equivalence classes of equinumerosity. Thus, from Frege's point of view, by completely fixing the references of value-range terms and by subsequently identifying cardinal numbers with value-ranges he has succeeded in uniquely fixing the references of numerical terms standing for cardinals as well. In *Frege 1893*, a Caesar problem does not arise from Hume's Principle, or to put it in slightly paradoxical terms: it is already solved before it could arise.

Frege badly needed Hume's Principle for laying the logical foundations of cardinal arithmetic. It was therefore imperative for him to establish this principle as a truth of logic, whatever the cost. It could seem that in the period after 1890 and before facing Russell's discovery, he had just two options to accomplish this: (1) To derive Hume's Principle from an explicit definition of the cardinality operator whose *definiens* was couched in purely logical terms; (2) to treat this principle as a logical axiom governing the cardinality operator as a primitive term of the formal language. Deriving Hume's Principle from his explicit definition of the cardinality operator is what Frege actually does both in *Grundlagen* and in *Grundgesetze*. At that time, this option was by far the most promising and most practicable. It did not require that Hume's Principle be self-evident. Moreover, it enabled Frege (a) to provide a deductive justification for it and (b) to secure its requisite status as a truth of logic. Note that after 1890 his original tentative usage of Hume's Principle as a contextual definition of the cardinality operator was no longer an option for him, quite independently of the impact that the Caesar problem had on the acceptability of the tentative contextual definition of "$N_x\varphi(x)$". After 1890, Frege relied on a theory of definition for his concept-script with strict principles prohibiting any definition of this kind. Contextual definitions offend against his principle of the simplicity of the *definiendum* and possibly against his prohibition on piecemeal definitions as well.

Not surprisingly, due to Russell's paradox even the first option was no longer available to Frege, unless he had succeeded in contriving an explicit definition of the cardinality operator whose *definiens* was arguably couched in purely logical terms, but did not rest on an *inconsistent* theory of extensions of concepts or even on extensions of concepts at all — a possibility that appears remote, but not absurd. As to the second, more appealing option in the face of Russell's paradox, there is not even a trace of evidence in Frege's letter to Russell, nor anywhere else in his post-contradiction work, that Frege was toying or even wrestling with it. This is not to rule out, however, that he reflected on this option and had it in mind when he wrote the letter to Russell.

Elsewhere (in *Schirn 2006*) I have argued at length that in the light of the constraints that Frege imposes on the acceptability of a given thought as a primitive law of logic he could hardly have chosen Hume's Principle as a logical axiom governing the cardinality operator as a primitive function-name of the formal language. I presented several reasons for this — perhaps of slightly different weight — but the threat of inconsistency was definitely not one of them. Suffice it here to mention three of the reasons.

Firstly, Hume's Principle does not hold with unrestricted generality and is not universally applicable, because it cannot be made true in any finite domain. Its existential quantification is false in every finite domain. If n objects belong to the domain, there have to be $n + 1$ different cardinalities. Yet according to Frege's conception of logic, utmost generality is a *conditio sine qua non* for a true proposition to be acknowledged as a primitive law of logic.[35]

Secondly, if Frege were to assign to Hume's Principle qua expression of an axiom of a theory T the role of fixing (at least partially) the reference of the cardinality operator construed as a primitive expression in the language of T, he would flagrantly offend against his own tenet that it can never be the task of the expression of an axiom to fix the sense and the reference of a sign, especially of one occurring in it. For if the latter were the case, then in Frege's view Hume's Principle would not express a thought at all, (consequently) lack a truth-value and, by the same token, not express an axiom (cf., for example*, Frege 1967*, p. 283; *Frege 1976*, p. 62).[36]

A third reason is what in section 6 I shall term *Frege's dilemma* and discuss in detail: If Frege had installed Hume's Principle as an axiom of his formal theory, he would have faced a head-on conflict between the requirements of self-evidence on the one hand and genuine knowledge or real epistemic value on the other.

Setting these doubts aside for a moment, imagine that in the aftermath of Russell's paradox Frege thought that he could make a virtue of necessity and play his last trump to avoid the contradiction and salvage the idea of logicism. More specifically, suppose that he thought he could introduce the cardinality operator as a primitive second-level function-name by means of a stipulation modelled upon the pattern of the semantic stipulation in *Grundgesetze,* §3 regarding value-ranges which I shall quote and consider at the beginning of the next section:

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[35] I owe this insight to a private discussion with George Boolos in the summer of 1993 about the status of Hume's Principle. See in this connection *Boolos 1997*, p. 255. An anonymous referee pointed out to me that given "that Frege did accept Basic Law V as a law of logic and given that Basic Law V also cannot be made true in any finite domain", he or she believes that we are facing a problem here. I basically agree. If, due to its lack of maximal generality, Hume's Principle is not a proper candidate for being regarded as a primitive truth of logic from Frege's point of view, then the same argument applies to Basic Law V. And from this observation it would follow that by his own lights Frege would have been ill-advised to choose Basic Law V as the key axiom of his logicist project, even independently of the problem that it lacked the requisite (degree of) self-evidence. You will recall my earlier statement that if n objects are in the domain, Axiom V requires the existence of 2^n abstracta. So, Basic Law is clearly inflationary.

[36] In his lecture 'Über formale Theorien der Arithmetik', Frege seems to rule out an axiomatic introduction of the cardinality operator, although he does not mention this term. He puts forward the requirement that everything arithmetical be reducible to the logical by means of definitions (*Frege 1967*, p. 104).

"I use the words 'the concept $\Phi(\xi)$ has the same cardinal number as the concept $\Psi(\xi)$' generally as coreferential with the words 'the concepts $\Phi(\xi)$ and $\Psi(\xi)$ are equinumerous'." Henceforth, I refer to this hypothetical stipulation as "SC".[37] Suppose also that at the outset of the exposition of the concept-script Frege had temporarily narrowed down the first-order domain of his logical theory to the True and the False and cardinal numbers, being aware that the logical development of real analisis and of the arithmetic of complex numbers at a later stage in his foundational project would require certain extensions of the domain. Suppose further that he had succeeded in removing the referential indeterminacy of the cardinality operator — arising inevitably from SC despite the assumed limitation of the domain — in a fashion similar to the strategy that he pursues in *Grundgesetze*, §10 with the purpose of fixing completely the reference of "$\grave{\varepsilon}\varphi(\varepsilon)$". By appealing to a special permutation argument along the lines of the permutation argument that he presents in §10 and by subsequently invoking a related identifiability thesis: "Thus, without contradicting our equating '$N_xF(x) = N_xG(x)$' with '$Eq_x(F(x),G(x))$', it is always possible to determine that an arbitrary cardinal number be the True and another arbitrary cardinal number the False", Frege could have felt entitled to identify the number 1 with the True and the number 0 with the False, for example. Finally, suppose — again for the sake of argument — that he thought he had a sound argument for the logical nature of Hume's Principle as an axiom and accordingly laid it down as an axiom of his formal theory.[38]

For obvious reasons, it would be grossly misleading to say that in such a situation Hume's Principle could or would *replace* the discredited Axiom V. Axiom V was designed to introduce logical objects of a fundamental and irreducible kind with which all numbers, not only the cardinals, had to be identified in order to justify their assumed purely logical nature. After having accomplished this, Axiom V could have been regarded, from Frege's point of view but prior to Russell's discovery of the contradiction in *Grundgesetze*, as a means that affords us the appropriate cognitive access to numbers of all kinds and therefore provides the key to a uniform answer to the question "How do we grasp the numbers?". It is hereby presupposed that by virtue of his additional stipulations in §§10-12 he had succeeded in

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[37] Recall that in Frege's view the expression of an axiom cannot fix the reference (and the sense) of any expression, let alone the reference of an expression occurring in it. Fixing the reference of a term is the exclusive task of an elucidation or a definition or of a special semantic stipulation (cf. *Grundgesetze*, §3) that differs from both a standard elucidation of a function-name and a formal explicit definition.

[38] The idea that I am describing was first developed in *Schirn 1996*, pp. 168-169. I present it here in a different form.

endowing each canonical value-range term with a unique reference. By contrast, the power and efficiency of Hume's Principle qua stipulation SC or qua axiom emerging from SC would of course have been restricted to the introduction of cardinal numbers as logical objects and to providing the means of apprehending them. Thus, concerning the real and complex numbers, it would have been imperative for Frege to find appropriate means — possibly logical abstraction principles — that do for these numbers what in the scenario I am imagining Hume's Principle was designed to do for the cardinals: introducing a suitable primitive number operator by laying down identity conditions for the real or complex numbers in terms of a second-order (or higher-order) equivalence relation.

I close this section by adding a final brushstroke to the picture I have painted so far.

Neither in *Grundlagen* nor in *Grundgesetze* did Frege see any need to raise the question of whether Hume's Principle is self-evident or not. The most likely reason for this is that in the two works he did not select Hume's Principle as a logical axiom, but tried to establish it as a truth of logic by way of deductive proof. If he had been asked "Do you regard Hume's Principle as self-evident?", he perhaps would have replied: "If I did, I could have spared myself the trouble of proving it, since if it were self-evident, it would not need deductive justification." For Frege, Hume's Principle is the fulcrum of the proofs of the basic laws of number theory. If he considered this principle to be self-evident — and this would imply that he regarded its two sides as synonymous and thus the principle itself as an epistemic triviality — it would remain unfathomable how he could have intended to derive the whole wealth of cardinal arithmetic from a truth that expresses the same thought and has the same epistemic value as "$N_xF(x) = N_xG(x) \leftrightarrow N_xF(x) = N_xG(x)$" or "$Eq_x(F(x),G(x)) \leftrightarrow Eq_x(F(x),G(x))$".[39]

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[39] Frege's proof of Hume's Principle (sentence 32) in *Frege 1893*, central as it is for laying the logical foundations of cardinal arithmetic, proceeds in six stages and is fairly complex. Like several other proofs in this volume, it did not fall into his lap. Each construction (*Aufbau*) (see §§ 55, 57, 59, 61, 63, 65) is preceded by what Frege terms "analysis" (*Zerlegung*). The force of the proof is to be sought only under the heading "construction". Regarding the details of Frege's formal proof of Hume's Principle see *Schirn 2016* and *May and Wehmeier 2017*.

In *Frege 1893*, §50, Frege comments on "$a = a$" qua theorem of his logical calculus. He writes: "Although this sentence is by our explanation of the equality-sign obvious [*selbstverständlich*], it is nonetheless worth seeing how it can be developed out of (III)." (III) is Basic Law III: $g(a = b) \to g(\forall f(f(\mathrm{a}) \to f(b)))$, in words: the truth-value $\forall f(f(\mathrm{a}) \to f(b))$ falls under every concept under which the truth-value $a = b$ falls. "$a = a$" does not need proof, because it is obvious. Deducing "$a = a$" nevertheless from Basic Law III is not pointless, although the proof does not amount to furnishing a deductive justification for it. This tallies with Frege's remark that it is worth the effort to show how "$a = a$" can be inferred from Basic Law III, despite the obviousness of this sentence in the light of his explanation of the equality

4. Axioms in general and Axiom V in particular: the requirement of self-evidence

The mark of distinction that general primitive truths are supposed to wear on their sleeve is their self-evidence combined with real epistemic value. Thanks to their self-evidence, they do not need proof to be acknowledged. Of course, once they are selected as axioms of a theory *T* they do not even admit of proof in *T*.

This is unquestionably Frege's view.[40] It is unfortunate that he never explained why he thought that Axiom V lacks the required (degree of) self-evidence, that is, why he believed that it is not of itself immediately evident, from the sense of its expression.[41] I presume that in his opinion the lack of the requisite (degree of) self-evidence of Axiom V had first and foremost to do with the semantics of its formal expression. It is true that in *Frege 1893*, §3 Frege stipulates only that the sentence (combination of signs) expressing the coextensiveness of two monadic first-level functions *f* and *g* shall be coreferential with the sentence expressing the identity of the value-ranges of *f* and *g*. I call this semantic stipulation concerning the metalinguistic analogue or counterpart of "$\grave{\varepsilon}\varphi(\varepsilon)$" *contextual stipulation* because it is immediately reminiscent of the attempted contextual definition of the direction operator in *Grundlagen*, §65, which in fact reads very similarly. This stipulation, which is later embodied in the formal version of Axiom V (cf. *Frege 1893*, §§9, 20)[42], is non-standard because, unlike the elucidations of the other primitive function-names of Frege's system, it does not directly

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sign. The proof even of an obvious truth in a theory *T* may serve to gain a deeper insight into the inferential links that exist between the truths of *T* and, hence, into the logical structure of *T*. In 'Logik in der Mathematik', Frege argues in this vein: "A proof does not only serve to convince us of the truth of what is proved; it also serves to reveal logical relations between truths. This is why Euclid already proved truths that appear to need no proof, because they are evident without one" (*Frege 1969*, p. 220). See also *Frege 1884*, §2 and especially the remark in *Frege 1969*, p. 171 on what constitutes the value of mathematical knowledge. Now as far as Frege's proof of Hume's Principle in *Frege 1893* is concerned, I do not have the slightest doubt that he regarded it as a deductive justification. Especially in the light of the length and complexity of the proof, it seems unlikely that he took this principle to be obvious.

[40] See, for example, *Frege 1967*, pp. 263, 265. On p. 265, Frege writes that it is undoubtedly the case with axioms in the traditional sense of the word that real knowledge is contained in them. It is likewise clear that he always expressly endorsed the classical Euclidean conception of axioms and declared this conception to be sacrosanct.

[41] Cf. *Frege 1967*, p. 393: "The assertion of a thought which contradicts a logical law can indeed appear, if not nonsensical, then at least absurd; for the truth of a logical law is of itself immediately evident, from the sense of its expression."

[42] Someone might wish to argue that owing to this fact Axiom V has a quasi-stipulative character. Frege could scarcely accept this since in his view it can never be the task of an axiom to stipulate anything, let alone define, an expression.

assign a reference (and a sense) to the name of the value-range function — here it is initially only the informal counterpart of "$\grave{\varepsilon}\varphi(\varepsilon)$" — by stating the values that this function receives for fitting arguments, in this case for monadic first-level functions as arguments. It is rather designed to fix at least partially the reference of the name of that function by licensing the mutual transition from one mode of speaking which involves that name to another which does not (cf. in this connection Frege's wording in *Frege 1893*, §9 and *Frege 1903*, §146).[43] On the face of it, it could seem that it is this special mode of introducing the name of the value-range function that prevents Frege from showing that Axiom V is indubitable or self-evident by following the pattern that he uses when he argues for the incontestableness of his other axioms. Unfortunately, the matter is far from being perspicuous, and this is mainly due to the fact that Frege confines himself to giving only a meagre explanation, especially when he comes to present the concept-script version of Axiom V in *Frege 1893*, §20. Nowhere in this book does he raise the issue of undeniableness or self-evidence for that axiom, although he must have been aware that his logicist project may stand or fall depending on how this issue is settled.

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[43] If the contextual stipulation in §3 were expressly intended as a means of fixing the reference of a value-range term completely or uniquely, it would probably arouse suspicion from the very outset, due to the similarity it bears to the tentative contextual definition of the cardinality operator via Hume's Principle in *Frege 1884*. The criterion of identity for value-ranges, namely the coextensiveness of the corresponding functions, takes care of the truth-conditions of only those equations in which both related terms are canonical value-range names; I called those equations *canonical value-range equations*. Yet the criterion is powerless to determine the truth-conditions of "$\grave{\varepsilon}\Phi(\varepsilon) = q$", if "$q$" is not of the form of "$\grave{\varepsilon}\Psi(\varepsilon)$". In *Frege 1893*, §10, Frege proposes to achieve a more exact specification of value-ranges, that is, to remove the referential indeterminacy of value-range terms arising from the contextual stipulation in §3, by carrying out what I termed crudely "the procedure of function-value determination". At the stage of §10, the procedure boils down to determining the values of the identity relation $\xi = \zeta$. Somewhat surprisingly, Frege confines himself to determining its values only for value-ranges and the True and the False as arguments, contrary to what his phrase "just as for all other arguments" seems to suggest and also contrary to the fact that he takes the first-order domain of his logical theory to be all-encompassing. There is evidence for this fact, for example, in *Frege 1893*, §34, where he undeniably defines the dyadic membership function for all possible objects as arguments, that is, for an all-embracing domain; see in addition *Frege 1893*, §2; *Frege 1903*, §65 and the discussion in *Schirn 2016*, where I also argue that Frege's elucidations of the primitive first-level function-names of his system rest on the assumption that the first-order domain is all-inclusive. In any event, the contextual stipulation in §3 does not enable us to decide whether or not the True or the False is a value-range — hence the emergence of a variant of the old Caesar problem in *Frege 1884*, now clad in formal garb. In §10, Frege analyzes it and then offers a solution; see the discussion, for example, in *Heck 1999*, *Wehmeier* and *Schroeder-Heister 2005* and *Schirn 2017*, *2017b*.

Before I turn more closely to some of the problems surrounding Axiom V with special emphasis on the lack of the required (degree of) self-evidence, it will be useful to cast a glance at Frege's method of introducing the other axioms of his logical calculus by paying special attention to their claimed indisputableness. By way of comparison, the relevant features of Axiom V, especially concerning the requisite (degree of) self-evidence, will then appear in sharper outline.[44]

In his *Begriffsschrift*, Frege introduces the first axiom of his propositional calculus $|\!-a \to (b \to a)$ as follows: "The case in which a is denied, b is affirmed, and a is affirmed is excluded. This is self-evident [*Dies leuchtet ein*], since a cannot at the same time be denied and affirmed." Here Frege explains the self-evidence of this fundamental logical law by appealing to the semantic explanation of the conditional stroke given previously and the principle of non-contradiction of classical two-valued logic. The introduction of this axiom in *Frege 1893*, §18 proceeds basically in a similar manner. Note that Frege now employs the modal term "impossible" instead of "self-evident". According to the elucidation of the conditional function in *Frege 1893*, §12, $a \to (b \to a)$ could be the False only if both a and b were the True while a was not the True. "This is impossible; hence $|\!-a \to (b \to a)$" (Axiom I, §18). *This is impossible* — I take this phrase to mean here that the claim "$a \to (b \to a)$ is the False" could not be made without offending against the principle of non-contradiction; thus, $a \to (b \to a)$ is considered to be a necessary truth. In one place (*Frege 1969*, p. 267), Frege says expressly that an axiom must be necessarily true. I presume that this characterization is meant to apply, although perhaps not exclusively, to logical axioms.

Frege's introduction of the Axioms IIa, IIb, III and IV is very much akin to his presentation of Axiom I. As to Basic Law VI: $|\!-a = \backslash\dot{\varepsilon}(a = \varepsilon)$, governing the definite description operator "$\backslash\xi$" ("the substitute for the definite article"), he confines himself to stating succinctly that it follows from the *Bedeutung* (reference) of "$\backslash\xi$". According to the elucidation of "$\backslash\xi$" (see *Frege 1893*, §11), Axiom VI is the thought that every object Δ is identical with the value that $\backslash\xi$ has for the value-range $\dot{\varepsilon}(\Delta = \varepsilon)$ as argument.[45] I mention in passing that when in §11 Frege elucidates "$\backslash\xi$" he seems to presuppose tacitly that the reference of "$\dot{\varepsilon}\varphi(\varepsilon)$" has already been completely determined. However, it is only in §12

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[44] On the epistemology of Frege's basic laws of logic see *Pedriali 2017*.

[45] Frege employs "$\backslash\xi$"only once, namely when he comes to define the name of the "membership-function" in §34. It is the latter, not the former, that does essential work in the proofs of the theorems of *Grundgesetze*.

where he turns to the last of his five primitive first-level function-names — to the name of the conditional function — that the step-by-step process of fixing the reference of "$\grave{\varepsilon}\varphi(\varepsilon)$" appears to have come to an end; and Frege of course knew this. I have more to say on Axiom VI in section 6.

In Frege's view, propositional evidence implies that the truth of a given thought is beyond rational doubt and can be acknowledged instantly in a non-inferential way. In 'Logik in der Mathematik' (*Frege 1969*, p. 221) he writes:

> The *axioms* are truths as are the theorems, but they are truths which are not proved in our system, and which do not need proof. It follows from this that there are no false axioms, and that we cannot acknowledge a thought as an axiom if we are in doubt about its truth; for then it is either false and, therefore, is not an axiom, or it is true, but stands in need of proof and, hence, is not an axiom.

Setting Axiom V aside for the moment, I think that Frege could have summed up his method of introducing the axioms of his calculus as follows: For everybody who has followed my elucidations of the primitive function-names, the truth of the thoughts that I chose as axioms is obvious. Once a determinate reference and a determinate sense have been bestowed upon the primitive function-names of my logical system, each of which occurs in the expression of an axiom (in some cases in more than one), it is impossible to reject as false a thought that I singled out as an axiom; it is impossible because any such rejection would be self-contradictory or absurd. In short, my axioms are incontestable or evident from the sense of their expression (which is a name of the True or, when "|—" is prefixed to it, a *Begriffsschriftsatz*, a concept-script sentence) and, hence, from the senses of the component expressions and the way these are combined to form the name of the True (or the corresponding concept-script sentence).

Let us return to Axiom V. In *Frege 1893*, §9, Frege points out that the possibility of transforming the generality of a function-value equality into a value-range identity, and *vice versa* (§3), must be regarded as a logical law upon which the entire calculating logic of Leibniz and Boole rests.[46] A little later he makes another stipulation concerning "$\grave{\varepsilon}\Phi(\varepsilon)$"

[46] Cf. *Leibniz 1875-1890*, vol. 7, pp. 238-240. In what follows, "*A*" and "*B*" shall stand for concepts. According to Leibniz, we have: If $A = B$, then "*A* is in *B*" and "*B* is in *A*". One of two coinciding concepts is in the other. On p. 240, he states the converse: If "*A* is in *B*" and "*B* is in *A*", then $A = B$. Concepts which stand in the relation of mutual inclusion to one another coincide. Under an extensional interpretation of Leibniz' logic of concepts we obtain: $E(A)$ and $E(B)$ (that is, the extensions of *A* and *B* respectively) coincide if and only if $E(A) \subseteq E(B)$, and conversely $E(B) \subseteq E(A)$. In this connection, see also *Frege 1969*, pp. 16 f. where Frege appeals to Boole (whose chief novelty, however, was his theory of elective functions or,

which bears a certain similarity to the standard elucidations of the other primitive function-names: Generally speaking, “$\grave{\varepsilon}\Phi(\varepsilon)$” shall refer to the value-range of the function $\Phi(\xi)$. Just as the elucidation of the first-order quantifier “@a@ φ(a)” in §8 requires a supplementary stipulation settling the question of what the corresponding function $\Phi(\xi)$ is in each case, so too does the stipulation in §9. Yet the details of the supplementation need not concern us here.

At the outset of Frege *1893,* §10, Frege emphasizes that by having presented the combination of signs “$\grave{\varepsilon}\Phi(\varepsilon) = \grave{\alpha}\Psi(\alpha)$” as coreferential with “@a@ Φ(a) = Ψ(a)” he has not yet completely fixed the reference of a name like “$\grave{\varepsilon}\Phi(\varepsilon)$”. The stipulation concerning “$\grave{\varepsilon}\Phi(\varepsilon)$” in §9 is passed over in silence. I do not find this surprising since the intention underlying §9 is to introduce the notation for value-ranges in a practicable manner, not to fix the reference of “$\grave{\varepsilon}\Phi(\varepsilon)$”. In particular, it is the transformation of the generality of a function-value equality into a value-range identity, and *vice versa*, that must be expressible in Frege’s formal language. Let us assume that he had intended to place the stipulation in §9 concerning “$\grave{\varepsilon}\Phi(\varepsilon)$” on a par with the elucidations of the other primitive function-names. In this case, the stipulation should have conferred a definite reference on “$\grave{\varepsilon}\varphi(\varepsilon)$” at once. Consequently, the

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as we would put it today, his theory of truth-functions and their expression in disjunctive normal form). Frege would not endorse Leibniz’s claim that concepts which stand in the relation of mutual inclusion to one another coincide. According to him, concepts of whatever level cannot stand in the relation of identity to one another, since he takes it to be of first level, to hold only between objects. Yet he stresses that there is a close link between identity and the second-level relation of mutual subordination or coextensiveness between first-level concepts. Potter (2000, pp. 114 f.) contends that in Frege’s view concepts are identical if and only if they are coextensive. Yet this is incorrect. It is the extensions of *F* and *G* and not *F* and *G* themselves that are claimed to be identical if and only if *F* and *G* are coextensive; for Frege identity is a first-level relation. Writing “$F = G$” (or “$f = g$”) in his formal language — where “*F*” and “*G*” are schematic letters for monadic first-level function-names — is therefore illicit (see *Frege 1893*, §147; *Frege 1969*, p. 131; *Frege 1976*, pp. 197f.).

As to Leibniz’s logic, it is worth noting that he discovered an “arithmetical semantics” only for syllogistic logic, not for the more extensive *calculus universalis* of a general logic of concepts. In this semantics, he assigns pairs of numbers to the concept constants and interprets the operators belonging to the logic of concepts through certain arithmetical operations. As contentual elements of the *calculus universalis* he has: (a) concept constants *A*, *B*, *C*,...; (b) the operators of negation “non” and conceptual conjunction *AB*; (c) the relations of inclusion and identity as applied to concepts and their negations: $\subset$, $\not\subset$, $=$, $\neq$ (rendered as “est”, “non est”, “sunt idem” (or “eadem sunt”), “diversa sunt”) as well as the conceptual operator *M*(*A*) (“*A* est ens”, “*A* est possibile”) which is designed to single out in the set of concepts the consistent ones. It is noteworthy that as early as in his opus *Generales Inquisitiones de Analysi Notionum et Veritatum* of 1686 (see *Leibniz 1903*, pp. 356-399) Leibniz had found a complete axiomatization of the *calculus universalis* which is isomorphic to the standard set-theoretic algebra. Between 1686 and 1690 the *calculus universalis* had undergone a certain extension, thanks to the development of a theory of indeterminate concepts, conceived of as a “quantificational” extension of the algebra of concepts.

contextual stipulation in §3 would be dispensable with respect to fixing the reference of "$\grave{\varepsilon}\varphi(\varepsilon)$".[47] Furthermore, at the end of §10 Frege signals that he is pursuing a dual strategy to be carried out each time at one fell swoop, as it were: (a) further specifying the value-ranges and at the same time (b) determining the primitive first-level functions that must still be introduced for the purpose of laying the logical foundations of arithmetic (and are not reducible to the functions already known) by stipulating what values the latter should have for the former as arguments. If my previous assumption were to apply, then the stipulation in §3 would be redundant for the purpose of fixing the reference of "$\grave{\varepsilon}\varphi(\varepsilon)$". Moreover, any further specification of value-ranges following the strategy would likewise appear superfluous, although it could of course not be detached from the determination of the new primitive first-level functions. (a) and (b) are only two sides of the same coin, which is the act of stipulating what values a primitive first-level function should have for value-ranges as arguments. As soon as the references of canonical value-range terms are fixed uniquely, no further specification of the value-ranges is required.

A glance at §§10-12, 20 shows that the exact opposite of the assumption I made above is true. In Frege's view, it is only by stating identity conditions for value-ranges via the stipulation in §3 that the reference of a value-range term "$\grave{\varepsilon}\Phi(\varepsilon)$" is determined, albeit only incompletely as his line of argument in §10 makes clear.[48] Specifying value-ranges further by

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[47] If for Frege a sound elucidation of "$\grave{\varepsilon}\varphi(\varepsilon)$" had been feasible, that is, one which did not rest on a presupposed acquaintance with value-ranges, then he could have defined straight away the predicate "a is a value-range" ("$VR(a)$"), modelled on his definition of "n is a cardinal number" in *Frege 1884*, §72 (cf. *Schirn 1994*):

$$VR(a) := \exists\varphi(\grave{\varepsilon}\varphi(\varepsilon) = a).$$

Equipped with this definition, which, let us suppose, satisfies Frege's principle of completeness, he would have been in a position to decide, in principle, for every given object a whether or not it is a value-range. If a *is* a value-range and is given to us as such, Axiom V would tell us whether a is identical with a value-range b referred to by a canonical value-range name, that is, a term which is formed by inserting a monadic first-level function-name into the argument-place of "$\grave{\varepsilon}\varphi(\varepsilon)$". Unfortunately, the prospect for devising an irreproachable elucidation of "$\grave{\varepsilon}\varphi(\varepsilon)$" along the lines of Frege's elucidations of the other primitive function-names of his system were not encouraging for him. However, if I am right, then he would have agreed with the following diagnosis: if he had in fact succeeded in fixing completely the reference of "$\grave{\varepsilon}\varphi(\varepsilon)$" in the piecemeal fashion I characterized earlier, nothing would stand in the way of defining the predicate "$VR(a)$" in a section following section 12 of *Frege 1893*.

[48] In *Grundlagen*, §62, Frege says quite generally that we need a criterion of identity whenever we want to make sure that a singular term "a" refers to an object. If we consider the identity conditions on the right-hand side of Axiom V, we see that they are more tightly woven than those on the right-hand side of Hume's Principle. Plainly, the coextensiveness of

applying the strategy serves the purpose of fixing completely the reference of the name of the value-range function and thus of removing its initial referential indeterminacy. When in *Frege 1893*, §20 Frege presents Axiom V in a formal guise, it is essential that by virtue of his previous stipulations he has in fact succeeded in conferring a complete reference to "$\grave{\varepsilon}\varphi(\varepsilon)$". Otherwise, the formal expression of Axiom V would lack a determinate reference (truth-value) and, hence, would not express an axiom at all. It is of course likewise essential that the references of the other two function-names that occur in the formal expression of Axiom V have been fixed previously.[49]

In *Grundlagen*, Frege already makes it clear that it is a key prerequisite for the introduction of abstract or logical objects to lay down, in the first place, a general criterion of identity for them.[50] In §104, he deals briefly with fractions, irrational numbers, and complex numbers.

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two concepts F and G implies the equinumerosity of F and G — $\forall x(F(x) \leftrightarrow G(x)) \rightarrow Eq_x(F(x),G(x))$ — and by virtue of Hume's Principle also $N_xF(x) = N_xG(x)$, but the converse does not hold.

[49] Note that the senses of the four primitive function-names "$—\xi$", "$\xi = \zeta$", "@a@ φ(a)" and "$\grave{\varepsilon}\varphi(\varepsilon)$" are parts of Axiom V qua thought. "$\grave{\varepsilon}\varphi(\varepsilon)$" figures as the key term in the formal expression of that axiom. If one of these names lacked a sense, the equation between a value-range identity on the left and the corresponding generality of a function-value equality on the right (without the prefix "|—") would not express a thought at all. As I said in section 1, the Roman function-letters "f" and "g" that Frege employs on both sides of "=" in the formal expression of Axiom V indicate monadic first-level functions; they belong to the formal object-language (cf. *Frege 1893*, §19). In the informal, metalinguistic stipulation in §3, he uses "$\Phi(\xi)$" and "$\Psi(\xi)$". Frege emphasizes (§5, footnote 3) that he uses the capital Greek letters "Γ" and "Δ" as if they were names referring to something (an object), without specifying their reference. He adds that they will not occur in the development of the concept-script, just as little as "ξ" and "ζ". As a matter of fact, not only "Γ" and "Δ", but also names like "$\Phi(\xi)$" and "$\Psi(\xi, \zeta)$" are only used in part I of *Grundgesetze* entitled "Exposition of the concept-script". Although to my knowledge Frege does not say anything specific about the status of "$\Phi(\xi)$" and "$\Psi(\xi, \zeta)$", he most likely treats them as names that are on a par with "Γ" and "Δ", that is, he uses "$\Phi(\xi)$" and "$\Psi(\xi, \zeta)$" as if they were names referring to something (to any monadic or dyadic first-level function) without stating their reference. On the role of Frege's auxiliary names in the concept-script see *Heck 1997*, *Linnebo 2004* and *Schirn 2016*.

[50] In his "Habilitationsschrift" *Rechnungsmethoden, die sich auf eine Erweiterung des Größenbegriffes gründen* (1874), Frege writes (p. 51): "Quite generally speaking, the process of addition is the following: we replace a group of things by a single one of the same kind. This gives us a determination of the concept of quantitative identity. If we can decide in every case when objects agree in a property, then we obviously have the correct concept of the property. Thus in specifying under what conditions there is a quantitative identity, we determine thereby the concept of quantity." On the face of it, the way Frege describes the envisaged definitional introduction of the concept of quantity is reminiscent of his attempt in *Grundlagen* to introduce a function-name by means of a contextual definition in terms of an abstraction principle. Unfortunately, in *Frege 1874* he fails to specify a criterion of identity for quantities. Nevertheless, the method that he proposes there might be seen as a kind of

Just as in the case of the cardinal numbers, here, too, he says, everything will in the end depend on the search for a judgeable content which can be transformed into an equation, whose sides are just the new numbers. And this amounts to saying that he must first fix the sense of a recognition-judgement (*Wiedererkennungsurteil*) for such numbers. The fact that in *Grundlagen* Frege eventually sets up an explicit definition of the cardinal number belonging to the concept *F* in terms of the extension of a concept, and later suggests pursuing a similar strategy for the "higher" numbers, is not at odds with his view that stating identity conditions can and must be considered a road to success, whenever the introduction of abstract or logical objects is on the agenda.

In §20, Frege presents Axiom V in the concept-script version and observes: "We saw (§3, §9) that a value-range equality can always be transformed into the generality of an equality, and *vice versa*."

As to the problem of why we should believe in the indubitableness of Axiom V, this is uninspiring. Unlike the elucidation of (the name of) the conditional function or the elucidation of the definite description operator, each of which is supposed to guarantee that the corresponding axiom (Axiom I and Axiom VI respectively) is unassailable on semantic grounds, the contextual stipulation in §3 falls short of providing compelling grounds for believing in the incontestableness of Axiom V. We know from Frege's initial assessment in §10 that his stipulation in §3 fails to fix the reference of "$\grave{\varepsilon}\varphi(\varepsilon)$" completely. We further know that he thinks he had solved the problem of referential indeterminacy of "$\grave{\varepsilon}\varphi(\varepsilon)$" before he comes to present the formal version of Basic Law V in §20. However, in §20 he refrains from saying that Basic Law V follows from the reference of "$\grave{\varepsilon}\varphi(\varepsilon)$" (in fact it does not). He likewise desists from employing the modal term "impossible" along the lines of explaining the evidence of Basic Law I (how could he do this?), and he does not say that Basic Law V is immediately evident from the sense of its expression.

Suppose that in *Frege 1893*, §3 Frege had stipulated that the sentence expressing the generality of an equality of function-values and the corresponding value-range equation not only refer to the same truth-value but also express the same sense. In that case, he might have wished to say, when introducing the concept-script version of Axiom V in §20, that this axiom is of itself immediately evident, from the sense of its expression. Yet we can at best speculate whether Frege thought by stipulating not only coreferentiality in §3, but also sameness of sense, that the referential indeterminacy of value-range names which he attempts

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forerunner of his later introduction of logical objects via the formulation of identity conditions for them.

to resolve in a first crucial step in §10 could have been avoided in the first place. I assume that he did not believe he could derive substantial benefit from stipulating sense identity as far as his central aim of fixing completely the reference of the value-range operator is concerned. So, even if Frege thought that the two sides of Basic Law V do express the same thought, this would not have compelled him to include sense identity in the contextual stipulation in §3. Be this as it may, despite the hypothetical or speculative character of dealing with these and related issues, I now want to discuss the question of whether Frege considered the two sides of Basic Law to be synonymous or not. However, I shall not discuss, with respect to Basic Law V, the two criteria of thought identity which Frege formulated in 1906. One criterion is framed in logical terms (cf. *Frege 1976*, pp. 105 f.), the other in epistemic (cf. *Frege 1969*, p. 213). In *Schirn 2014a* and *Schirn 2016*, I apply the criteria to Hume's Principle and conclude that its two sides might come out as expressing the same thought according to the first criterion, but might also be considered to express different thoughts according to the second. I think that my line of argument there could *mutatis mutandis* be transferred to Basic Law V.[51]

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[51] Regarding Frege's conception of thought identity, Michael Dummett has propounded two theses with an eye to Fregean abstraction principles which stand on shaky ground. If they were correct, then there would be no point in raising the question of whether the two sides of Hume's Principle or of Basic Law V express the same thought or not. In *Dummett 1973*, pp. 378 f., Dummett writes: "To say that the sense of a sentence is composed out of the senses of its constituent words is to say ... that we can grasp that sense only as the sense of a complex which is composed out of parts in exactly that way; only a sentence which had exactly that structure, and whose primitive constituents corresponded in sense pointwise with those of the original sentence, could possibly express the same sense. (Frege's notion of the senses of complex expressions thus tallies closely with Carnap's intensional isomorphism.)" These remarks are plainly at variance with Frege's thesis (A): Different sentences can express the same thought. (A) applies also to certain pairs of sentences which, under their standard interpretation, have significantly different syntactic/logical structures such as, for example, "The number 9 belongs to the concept *planet*" (or "There are nine planets") and "The number of planets = 9" (cf. *Frege 1884*, §57 and *Frege 1967*, p. 173). If thesis (A) did not hold, then, as Frege stresses, logic would be paralyzed; for its "task can hardly be performed without trying to recognize the thought in its manifold guises" (*Frege 1967*, p. 170 footnote 7). Furthermore, Dummett's claim flies in the face of thesis (B): A sentence and a thought which it expresses may be analyzed or divided or decomposed in distinct ways (cf. *Frege 1967*, p. 173; *Frege 1969*, pp. 203, 218). For whenever this is done for a sentence (p) and the thought expressed by it, this very thought can be construed as being composed or built up in different ways. So, contrary to what Dummett contends, Frege's notion of the sense of a complex expression has little if anything in common with Carnap's notion of intensional isomorphism. In *Dummett 1991*, pp. 295 ff., Dummett contends that there is more than a kernel of truth in the assumption that after 1891 Frege tacitly accepted the following principle (K): If a sentence (a) involves a concept that a sentence (b) does not involve, then (a) and (b) cannot express the same thought. However, there is textual evidence that Frege did not wholeheartedly subscribe to (K). Here is just one counterexample to (K): (a) "There is at least one square root of 4" and

5. A closer examination of the two sides of Basic Law V: identity or difference of sense?

Before turning to Basic Law V, I wish to make a few remarks on Frege's view about the semantics of his abstraction principles in *Grundlagen*.

In *Schirn 2014a*, section 4, I argue that in *Grundlagen* a contextual definition of a term-forming operator that presents itself in the guise of an abstraction principle is probably intended to stipulate that its two sides shall have the same judgeable content. If this appears plausible, and I think that it does in the light of the available evidence (see *Grundlagen*, §§62, 65, 104 and also the comments in Schirn 2010a, section 2), then we are perhaps entitled to say: When in *Grundlagen*, §65 Frege stipulates that the two sides of a given first-order abstraction principle *qua contextual definition* of a term-forming operator shall be "*gleichbedeutend*", what he has in mind from the point of view of his later theory of sense and reference is that the two sides shall express the same thought (*Frege 1976*, pp. 96, 120; cf. *Frege 1967*, p. 172; *Frege 1893*, p. X). Note that in his definitions of the relation of equinumerosity, the concept of cardinal number (§72), the successor relation (§76), the strong ancestral (§79), the weak ancestral (§81), and the concept of finite cardinal number (§83) Frege likewise uses the word "*gleichbedeutend*". This applies also to his tentative contextual definition of the cardinality operator as §65 already suggests by analogy and §106 makes definitely clear. From the fact that in one place in *Grundlagen*[52] Frege employs the word "*Bedeutung*" with respect to a singular term (his symbol "∞_1" for the smallest infinite cardinal number) most likely in the sense of "reference" (see §84) we cannot infer that in his definitions (where the two sides are propositions) he uses the word "*gleichbedeutend*" in the sense of "coreferential" as he does later in *Grundgesetze*.[53]

I hasten to add that in my view Frege's characterization of a particular first-order abstraction in *Grundlagen*, §64 in terms of distributing the content of "//" to line *a* and line *b* or in terms of splitting up a content in a way different from the original way is misguided. Fregean abstraction, correctly understood, has nothing to do with Frege's method of extracting function-names from more complex names by means of what I call gap

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(b) "The concept *square root of 4* is realized". Frege asserts that (a) and (b) express the same thought (cf. *Frege 1967*, p. 173).

52 There may be a few other instances, but I did not check this.

53 Note in this connection that in one of his letters to Husserl (*Frege 1976*, p. 96) Frege mentions that he would now — after having drawn the distinction between sense and reference — prefer to replace in several places in *Grundlagen* (§§97, 100-102) "*Sinn*" ("sense") by "*Bedeutung*" ("reference").

formation.[54] It is mainly for this reason that I hesitate to take §64 in addition to §62, §65 and §104 as evidence that Frege regarded the two sides of an abstraction principle *qua contextual definition* as synonymous, although the opening claim in §64, namely that the judgement that line *a* is parallel to line *b* can be construed as an identity, speaks probably in favour of content identity and definitely not against it. I suggest Frege should have said something like this in §64, assuming that "*gleichbedeutend*" was in fact intended to mean (judgeable) content identity: An abstraction principle embodies the transition from one mode of speaking (a) (= the equivalence on the right-hand side) to another (b) (= the identity on the left-hand side) involving the desired function-name (singular term-forming operator) which (a) does not contain, or shorter: in an abstraction principle *qua contextual definition*, one and the same content is presented in distinct ways by (a) and (b). This does not imply that once Frege had rejected the contextual definition of the cardinality operator and treated Hume's Principle as a provable theorem he continued considering its two sides to be synonymous. As I have argued in section 3, neither in *Grundlagen* nor in *Grundgesetze* could it have been his proper intention to set up the requirement that the two sides of Hume's Principle qua provable theorem shall be synonymous.[55]

I now turn to Axiom V. We know that in Frege's notation its expression is an equation of the form "*a* = *b*" and we further know that in *Grundgesetze*, §3 he stipulates only that the two truth-value names flanking "=" shall be coreferential. To be sure, this stipulation taken by

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[54] See the arguments in *Schirn 2014a*, section 9 and *Schirn 2016*, section 5. In *Grundlagen*, §70, Frege describes the true method of forming first-level concepts and relations through analysis of a judgeable content, which, despite first appearances, is completely different from Fregean abstraction. This method corresponds to the syntactic device in *Grundgesetze*, §26 of forming concept-script function-names by way of gap formation.

[55] Ebert (*Ebert 2016*, section 2) argues against the "standard" view according to which Frege adopted "abstraction synonymy" in *Grundlagen*. See also Dummett's analysis in *Dummett 1991*, chapter 14; I do not endorse it in every respect. I think that there are good reasons to refrain from drawing any conclusion from Frege's semantic treatment of abstraction in *Grundlagen*, §§64-65 — regardless of whether one argues for or against "abstraction synonymy" — with respect to his view about the semantic relation between the two sides of the contextual stipulation and/or Basic Law V in *Grundgesetze*. In *Grundlagen*, the question "abstraction synonymy or not?" concerns abstraction principles qua tentative contextual definitions, whereas in *Grundgesetze* Frege is dealing in the first place with a non-definitional stipulation in §3 and then with a basic law of logic. Moreover, as I shall point out in a moment in a specific context, it is perfectly possible that even in the short period between the publication of 'Funktion und Begriff' (1891) and the completion of the first volume of *Grundgesetze* Frege had changed his mind about the semantic relation between the two sides of Basic Law V.

itself does not rule out that he regarded these names as expressing the same sense.[56] It is perfectly conceivable that concerning the question of whether the two sides of the contextual stipulation or of Basic Law V are synonymous, Frege vacillated between the pros and cons and finally refrained from making up his mind. For he must have been aware that he was facing unpalatable consequences in either case. If he did make a choice in this matter and thought that he could adduce a sound argument for it, it would seem rather odd that all this is passed over in silence in *Grundgesetze*. As I indicated above, Frege apparently believed that he could succeed in fixing at least partially the references of value-range names by stipulating only coreferentiality in §3, without any consideration of sense. However, independently of the aim of endowing each canonical value-range name with a unique reference via the contextual stipulation plus some additional stipulations the need to ensure the requisite (degree of) self-evidence of Axiom V must have been another pressing issue from the very outset of his project in *Grundgesetze*. And if this issue could have been settled at all, it had to be done by considering the relation of the sense of (a) to that of (b) in Basic Law V.

There is another thing that I take to be obvious in this connection. If in *Frege 1893* Frege had been convinced that the two sides of the expression of Axiom V not only refer to the same truth-value, but also express the same thought, he could scarcely have believed that this axiom is not as evident as one would wish for a primitive law of logic. So, if we give credence to what he says about Basic Law V in the Afterword to *Frege 1903* with an eye to the Foreword to *Frege 1893*, we must assume that he had at least serious doubts that its two sides express the same thought.[57] Why did he withhold this from the reader of *Grundgesetze*? In any event, a cogent argument to the effect that despite the assumed difference of sense between (a) and (b) in Basic Law V the required (degree of) self-evidence of this law could be secured, was presumably not forthcoming for Frege. But why not relinquish the requirement of self-evidence for logical axioms in general and for Basic Law V in particular and replace it with a weaker condition that, unlike the exigency of self-evidence, would allow for a distinctness of sense of (a) and (b) in Basic Law V? At the end of section 6, I shall briefly discuss the question of whether Frege might have been in a position to endorse such an option without having to pay an intolerably high price.

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[56] Note that if Frege had stipulated sense identity in §3, he would not have offended against any of his semantic principles, contrary to what Dummett seems to suggest in *Dummett 1973*, pp. 378 f. and *Dummett 1991*, pp. 295 ff.

[57] Ebert's line of argument in *Ebert 2016*, section 3 seems to support some of the points I make in sections 5 and 6. We agree that there are strong reasons to interpret Frege as rejecting "Basic Law V synonymy".

Peter Simons argues in *Simons 1992* that in Frege's opinion the two sides of Basic Law V not only refer to the same truth-value but also express the same sense. At the beginning of *Frege 1893*, §10, Frege claims that, on the assumption that $X(\xi)$ is a bijection of all objects (of the first-order domain of his logical theory), "$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$" is coreferential with "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$". You will of course recall that prior to this argument he had already stipulated (in §3) that "$\grave{\varepsilon}\Phi(\varepsilon)) = \grave{\alpha}\Psi(\alpha)$" and "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$" shall have the same reference. As regards the claim that "$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$" is coreferential with "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$", Frege adds in a footnote: "That is not to say that the sense is the same." This remark is almost trivial, since sameness of reference does not imply sameness of sense. The converse does apply, at least in a logically perfect language where every well-formed name is supposed to have a (unique) reference, not only a sense. According to Simons (1992, p. 764), Frege's remark does suggest, though, that the sense of "$\grave{\varepsilon}\Phi(\varepsilon) = \grave{\alpha}\Psi(\alpha)$" *is* the same as that of "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$". I disagree. In the light of both the contextual stipulation in §3, in which Frege is only concerned with coreferentiality, and the remark in the footnote, in which he neither affirms nor gainsays that the sense of "$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$" coincides with that of "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$", Simons's thesis is ungrounded. Perhaps he is assuming that Frege thought, albeit tacitly, that "$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$" and "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$" express the same sense and concludes from this that Frege likewise construed "$\grave{\varepsilon}\Phi(\varepsilon) = \grave{\alpha}\Psi(\alpha)$" and "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$" as synonymous. If so, the conclusion is most likely false.

Suppose, for the sake of argument and in contrast to what I suggested above, that Frege regarded the two sides of Basic Law V not only as referring to the same truth-value, but also as expressing the same thought. In that case, he would probably have found himself compelled to deny that "$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$" and "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$" likewise express the same thought. The reason for this is that in "$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$" the function-name "$X(\xi)$" is supposed to contribute (essentially) to the sense of "$X(\grave{\varepsilon}\Phi(\varepsilon))$" and to that of "$X(\grave{\alpha}\Psi(\alpha))$", and thus also to the sense of "$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$".[58] If this is correct, and I believe that it

[58] To the best of my knowledge, in his entire work Frege gives only one explicit example of a predicate, namely "is true", whose sense is said to contribute nothing to the sense of sentences in which it occurs, for example, in sentences of the form "The thought that *p* is true" (cf. *Frege 1969*, pp. 271 f.). Since I take him to be mistaken here, he fails to give a single example of an expression (singular term or predicate or function-name) that occurs in a syntactically relevant position in a sentence, but has only a "non-contributory" sense (which in the light of his concluding remark in *Frege 1893*, §32 on the senses of subsentential expressions may sound like an absurdity). Here are the reasons. If the sense of "is true" did not contribute essentially to the sense of (*S*) "The thought that *p* is true", (*S*) would not express a thought at

is, then we must conclude that the sense of "$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$" differs from that of "$\grave{\varepsilon}\Phi(\varepsilon) = \grave{\alpha}\Psi(\alpha)$". Thus, on the assumption that I made above for the sake of argument, the sense of "$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$" would be distinct from that of "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$". Clearly, if Frege affirmed identity of sense in the case of "$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$" and "$\forall x(\Phi(x) \leftrightarrow \Psi(x)$", he would be committed to the claim that the two sides of the contextual stipulation or Basic Law V express different thoughts, contrary to what Simons asserts. But it is equally obvious that if Frege claimed that the sense of "$X(\grave{\varepsilon}\Phi(\varepsilon)) = X(\grave{\alpha}\Psi(\alpha))$" differs from that of "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$", he would not be forced to contend that the two sides of Basic Law V

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all, but only the sense of the complex singular term "the thought that *p*". This term *refers* to the thought expressed by "*p*", but it does not express it. It is "*p*" qua constituent of "the thought that *p*" — and of (*S*) — that expresses this thought. The functional expression "the thought that" clearly has a sense that likewise contributes essentially to the non-propositional sense of "the thought that *p*". And "is true" as a constituent of (*S*) does not miraculously absorb what seems to be a "surplus" sense from the point of view of Frege's claims (a) that the sense of "is true" contributes nothing to the sense of (*S*) and (b) that "*p*" and (*S*) express the same thought. On the contrary, the sense of "is true" does make an essential contribution to the sense of (*S*) just as the sense of "is interesting" or of "is appealing" contributes essentially to the sense of "The thought that *p* is interesting [appealing]". How else could the thought expressed by (*S*) have been "built up" from the senses of the parts of (*S*)? In accordance with a general thesis of Frege's about the homorphism between sentence-structure and thought-structure (cf. *Frege 1969*, p. 243, 262, 275; *Frege 1976*, p. 127), but contrary to what he contends in the case of (*S*), the structure of (*S*) can — figuratively or metaphorically speaking — indeed serve as a picture of the structure of the thought expressed by it. We have the subsumption of the thought that *p,* referred to by "the thought that *p*", under the concept *is true*, and in addition — what Frege seems to overlook or drop (cf. *Frege 1969*, p. 211) — the relation of the sense of (*S*) to its reference. Contrary to what he asserts, language does *not* mislead us here (cf. again p. 211), and contrary to what he seems to believe (see *Frege 1969*, p. 252), *is true* is in fact a property of (true) thoughts. All true thoughts are true, this is undeniable. That the True is not a property of a thought goes without saying. In short, Frege fails to advance a convincing semantic argument for the claimed synonymy of (*S*) and "*p*". Moreover, judgement and assertion, which he additionally invokes when presenting his argument for the alleged sense identity of (*S*) and "*p*" (see, for example, *Frege 1967*, p. 150), have no impact whatsoever on the semantics of a sentence; and he must have been aware of this. As far as I know, Frege nowhere discusses the question of whether the members of a pair of *truth-value names* — such names are in my use of this expression concept-value or relation-value names, that is object names which have the syntactic structure of a declarative sentence — where the second member is obtained by inserting the first into the argument-place of "$—\xi$", have the same sense or express different senses. (I disregard here the case where in such an operation the initial name is of the form "@Δ" or "#Δ". Such a name is only converted into itself by fusing the horizontals.) It is, however, obvious that whenever we transform an object name "Δ" of Frege's formal language, which is not a truth-value name, into a truth-value name "@Δ", the sense of "@Δ" differs from that of "Δ", even if the two names are coreferential. Thus, both names "$\grave{\varepsilon}(\varepsilon = (\varepsilon = \varepsilon))$" and "$— \grave{\varepsilon}(\varepsilon = (\varepsilon = \varepsilon))$" refer to the True by virtue of Frege's stipulations, but whereas the first has intrinsically a non-propositional sense, the second expresses a thought.

express the same sense. Finally, the assumption that "$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$" and "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$" express different thoughts does not imply that "$X(\dot{\varepsilon}\Phi(\varepsilon)) = X(\dot{\alpha}\Psi(\alpha))$" and "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$" likewise express different thoughts. The only combination that is definitely ruled out on the assumption I made above is that *both* "$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$" and "$X(\dot{\varepsilon}\Phi(\varepsilon)) = X(\dot{\alpha}\Psi(\alpha))$" express the same thought as "$\forall x(\Phi(x) \leftrightarrow \Psi(x))$". So, the little footnote to §10 does not suggest, let alone reveal anything about Frege's view regarding the identity or difference in sense of the two sides of Basic Law V. Let me add that regarding "$(X(\dot{\varepsilon}\Phi(\varepsilon)) = X(\dot{\alpha}\Psi(\alpha))) = (\forall x(\Phi(x) \leftrightarrow \Psi(x)))$" the question "self-evident or not?" and, hence, "sense identity or sense difference of the truth-value names flanking '='?" was not an issue for Frege since he did not use this formula as the expression of an axiom.[59]

In his letter to Russell of 28th July 1902, Frege observes that wherever the coincidence of reference is not obvious (*selbstverständlich*), we have a difference of sense.[60] Applied to the contextual stipulation of §3 this would mean: Assuming that in the course of writing the first volume of *Grundgesetze* Frege had in fact serious doubts about the requisite (degree of) self-evidence of the mutual transformation of the coextensiveness of two monadic first-level functions into a value-range identity, he hardly could have thought that prior to the contextual stipulation the coincidence of the reference of the sentence expressing coextensiveness with that of the sentence expressing identity is obvious and, hence (assuming that the converse of Frege's claim above holds), that the two sentences are synonymous. Since the contextual stipulation was not intended to stipulate the synonymy of the two sentences, the non-obviousness of the coincidence of reference did not cause a real problem for Frege — a stipulation need not stipulate an obvious or a self-evident fact. But one might perhaps argue that once the contextual stipulation was effectively made in §3, the coincidence of reference was then indeed obvious. So, when Frege comes to present the concept-script version of Basic Law V in §20, does he tacitly assume that its two sides not only refer to the same truth-value, but also express the same thought? If so, this would run counter to the initial doubts that he allegedly had about the requisite (degree of) self-evidence of Axiom V.

In 'Funktion und Begriff' of 1891 (*Frege 1967,* p. 130), Frege observes that "$x^2 - 4x = x(x - 4)$" expresses the same sense as the corresponding value-range equation "$\dot{\varepsilon}(\varepsilon^2 - 4\varepsilon) = \dot{\alpha}(\alpha\,(\alpha - 4))$", but in a different way.

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[59] The foregoing discussion is a revised and enlarged version of an argument that I presented in *Schirn 2006*, pp. 197 f.

[60] Thanks to Philip Ebert for drawing my attention to Frege's observation.

> It presents the sense, if we understand it as above, as the generality of an equation, whereas the newly introduced expression is simply an equation whose right side as well as its left has a complete reference in itself.

By appealing to this remark, Simons (1992) suggests with a rather mild proviso that “we take Frege at his word when he says that the two sides of (V) express the same sense but in different ways” (765). I do not think that we should interpret Frege’s remark as furnishing evidence for Simons’s claim that in *Frege 1893* Frege regarded the two sides of Basic Law V as expressing the same sense. It is, for example, perfectly conceivable that Frege had changed his mind in this respect, even though the first volume of *Grundgesetze* appeared only two years after the publication of ‘Funktion und Begriff’. In any event, what matters after all is the fact that in *Frege 1893*, §3 he does not refer to his earlier remark in ‘Funktion und Begriff’ *about an instance of Basic Law V*, and he does not use the notion of sense at all in §3.

Simons reminds us in connection with the quoted passage above that in ‘Über Sinn und Bedeutung’ Frege said that the sense of an expression (a proper name) “contains the mode of being given” of its referent. Simons goes on to observe (p. 764):

> On the face of it then, when saying the sense is given in more than one way, it would seem that Frege should have said that the referent is being given in more than one way, i.e. the two sides of (V) have different referents. But then if this is the most favourable case, unlike the one obtained by employing the *X* permutation, then the two sides of (V) necessarily differ in sense, and then it really is arbitrary which CV function we adopt.

I find this confusing. When in ‘Funktion und Begriff’ Frege says that one of the two sentences under consideration *presents* the sense as the generality of an equation, his mode of phrasing should in no way be conflated or lumped together with the explanation in ‘Über Sinn und Bedeutung’ regarding the sense of a proper name. In ‘Funktion und Begriff’, Frege simply intends to convey that a specific sense is expressed in one case as a generalization of an equality and in the other as an equality of value-ranges. Contrary to what Simons suggests, from this it does not seem to follow that Frege should have said that the two sides of Basic Law V have different referents, which would be absurd, nor that they express different senses. And it would likewise be inept to consider the possibility that in the context of §10 “$\dot{\varepsilon}\Phi(\varepsilon) = \dot{\alpha}\Psi(\alpha)$” and “$\forall x(\Phi(x) \leftrightarrow \Psi(x))$” have different references, while “$X(\dot{\varepsilon}\Phi(\varepsilon)) = X(\dot{\alpha}\Psi(\alpha))$” and “$\forall x(\Phi(x) \leftrightarrow \Psi(x))$” are coreferential.

In an undated letter to Peano, Frege gives an example of a first-order abstraction principle. He asserts that the sentence “The class of lines equal in length to *A* = the class of lines equal in length to *B*” expresses the same essential content as the sentence “The lines *A* and *B* are congruent". I presume that Frege wrote this letter after having completed *Frege 1893*. It seems that "the same *essential* content" is here not strictly tantamount to "the same thought". I therefore suggest that we should not infer from Frege's use of the first phrase that he construed the two sides of Basic Law as synonymous.[61] In my view, this would be unjustified, although I tend to assume that regarding the question "sense identity or sense difference of the two sides of an abstraction principle?" Frege did not distinguish between first-order and higher-order principles. Nonetheless, we cannot rule out that he drew such a distinction, nor can we definitely exclude that regarding second-order principles the question had to be answered separately on an individual basis.[62]

6. The choice of an abstraction principle as an axiom of a theory *T*: Frege's dilemma

My central thesis in this section is that in the light of his classical, Euclidean conception of axioms and due to the emphasis he places on the requirement of self-evidence of an axiom Frege faces an epistemic dilemma whenever he intends to choose an abstraction principle of the form "$Q(\alpha) = Q(\beta) = R_{eq}(\alpha, \beta)$" as an axiom of a theory *T*. Here "*Q*" is a singular term-

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[61] There is a similar remark by Frege on an instance of Basic Law V in 'Ausführungen über Sinn und Bedeutung' (*Frege 1969*, p. 132). He observes that on both sides of the instance under consideration one has expressed what is essentially the same thought. Strictly speaking and especially from a logical point of view, sense identity does not allow for degrees. Thus, it could appear doubtful to say that the two sides of an instance of Basic Law V express what is essentially the same thought (or almost the same thought).

[62] Sluga's analysis of Basic Law V (*Sluga 1980* and *Sluga 1986*) is fraught with errors, obscure formulations and terminological confusion. A detailed discussion of his views regarding Axiom V can be found in ***Schirn 2017a***. One problem in Sluga's exposition is that he fails to see that from Frege's point of view identity of sense of two coreferential singular terms "*a*" and "*b*" is only a sufficient, but not a necessary condition for the logical or analytic character of "*a* = *b*". This applies also to Basic Law V. In all likelihood, Frege did not think that Basic Law V is purely logical because its two sides express the same sense. Moreover, Sluga (1980 and 1986) ignores or overlooks the fact that Basic Law V boils down from both a semantic and an epistemic point of view to an instance of "*a* = *a*", if it is assumed that its two sides express the same thought. As I shall argue in a moment, the observation that the two sides of Basic Law V have significantly distinct syntactic structures lacks epistemic relevance once they are taken to be synonymous. Sluga's vague characterization of the notion of semantic content for the sake of explaining the semantic and epistemic nature of Basic Law V (see *Sluga 1986*) has no explanatory force, nor is it in the spirit of *Grundgesetze* where there is simply no room for a semantic concept to play an important role besides sense and reference. Sluga (1980) erroneously suggests that in the semantic stipulation in *Grundgesetze*, §3 "*gleichbedeutend*" is meant as "*sinngleich*". There is more terminological confusion in his account, for example, with respect to Frege's use of the term "(conceptual/judgeable) content" in *Begriffsschrift* and his use of "content" in one place of 'Funktion und Begriff' (*Frege 1967*, p. 126). Unfortunately, this has a negative impact on Sluga's entire exposition.

forming operator, α and β are free variables of the appropriate type, ranging over the members of a given domain, and "R_{eq}" is the sign for an equivalence relation holding between the values of α and β.[63] The dilemma applies especially to Axiom V, but it would equally apply to Hume's Principle, if Frege were to select it as an axiom of *T*, or to the choice of any other abstraction principle. I assume that according to Frege (a) the epistemic triviality of a thought goes hand in hand with its self-evidence, while (b) the converse does not hold generally. An axiom must be self-evident, but at the same time it should have real epistemic value or contain genuine knowledge (see, for example, *Frege 1967*, p. 263). The foundation of a fruitful axiomatic theory cannot consist of sheer trivialities. (c) If in an equation of the form "*a* = *b*" the expressions "*a*" and "*b*" not only refer to the same object, but also express the same sense, then "*a* = *b*" is epistemically trivial. "*a* = *b*" can be converted into an equation of the form "*a* = *a*" without any change of sense; and such an equation is undoubtedly trivial. It is, moreover, true by virtue of its form and therefore a truth of logic. This is of course not to say that any logical truth is trivial which, according to Frege, is clearly not the case. [64] (d) An equation of the form "*a* = *b*" to which (c) applies, is, according to (a), self-evident. (e) The self-evidence of a true equation of the form "*a* = *b*" implies that "*a*" and "*b*" express the same sense and, hence, according to (c) that "*a* = *b*" is epistemically trivial. (f) If in a true equation of the form "*a* = *b*" "*a*" and "*b*" express different senses, then "*a* = *b*" is neither self-evident nor epistemically trivial.

Now, the epistemic dilemma that Frege faces with respect to Axiom V is this. Almost trivially, he regards the two sides of its linguistic expression as coreferential because this is what he stipulates in *Frege 1893*, §3. If the two sides did not refer to the same truth-value, the claim that it is possible to transform one side into the other, and *vice versa* would be false and therefore not an axiom. (1) Suppose that Frege further holds that the two sides express different thoughts. In this case, he could justifiably assert that Axiom V meets the requirement of containing real knowledge, but at the same time he would have enormous trouble arguing for the necessary self-evidence of Axiom V qua axiom. (2) Suppose that Frege construes the two sides of Basic Law V as expressing the same thought. In this case, the required self-evidence of the axiom would be guaranteed, but it would be hard, if not hopeless to show conclusively that here self-evidence does not imply epistemic triviality. To point out that according to (2) one and the same thought is presented on the left side as an identity and on

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[63] We would of course replace the second occurrence of "=" by "$\leftrightarrow$".

[64] Note that it was explicitly presupposed that "*a*" has a reference. If "*a*" does not refer to anything, "*a* = *a*" is neither true nor false for Frege.

the right side as the generality of an identity does not invalidate the claim I just made. So, for Axiom V or for any other abstraction principle that is designed to figure as an axiom of a theory *T* the case in which both real epistemic value and self-evidence are given their due is ruled out. In other words, Frege can't have his cake and eat it. Abstraction principles seem to be under a curse for him as soon as he decides to pick them out as axioms of a theory *T*.[65]

Before I conclude this section with reflections on a possible way out of this impasse, let me consider an objection that has been raised to my previous line of argument. In 'Über die Grundlagen der Geometrie', I, 1903 (*Frege 1967,* p. 263), Frege stresses that even if what a definition has stipulated is subsequently expressed as an assertion, its epistemic value is not greater than that of an example of the law of identity $a = a$. He goes on to say rather cautiously:

> For although one could at best [*allenfalls*] call the law of identity itself an axiom, still one would hardly wish to assign the status of an axiom to every single instance, to every example of the law. For this, greater epistemic value is required.

From these remarks we cannot infer that Frege construed "$a = a$", in contrast to its instances, as a law containing genuine knowledge or real epistemic value. We may only conclude: if we could justifiably assign to "$a = a$" (or more precisely: to the thought it expresses) the status of an axiom, "$a = a$" could, in contrast to every instance, possibly be considered to possess real epistemic value. Yet we have already seen that in *Grundgesetze* Frege does not assign an axiomatic status to $a = a$. By his explanation of the equality-sign, $a = a$ is obvious (*selbstverständlich*), hence epistemically trivial, but he proves it nevertheless for reasons that I mentioned ealier.

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[65] I believe that the statement even in this general form resists refutation. To challenge it nonetheless would require that at least one counter-example be presented, that is, a Fregean abstraction principle which is self-evident although its two sides express different thoughts, or, alternatively, an abstraction principle that contains real knowledge although its two sides express the same thought. It does not really matter for my line of argument whether, say, a second-order abstraction principle appears in the guise of an equation modelled upon the pattern of the expression of Axiom V (or as a generalized equation) or as an equivalence (or as a generalized equivalence). Note that in *Frege 1893*, §53 Frege presents the formal version of Hume's Principle as follows: $v \in (u \in \rangle \int q) \rightarrow (u \in (v \in \rangle q) \rightarrow Nu = Nv)$. I use here "$\in$" for Frege's "membership-function" (§34), his symbol "$\rangle$" for the mapping-into by a relation (§38), the symbol "$\int$" for the converse of a relation (§39) and "$N\xi$" for the first-level cardinality function (§40).

The objection that has been raised to the argument I presented at the beginning of section 6 runs as follows. Thanks to the fact that Axiom V qua logical axiom is endowed with utmost generality, it can be considered to contain genuine knowledge even if we assume (recall assumption (2) above) that the two sides of its concept-script expression "$\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)$" and "$\forall x(f(x) = g(x))$" not only refer to the same truth-value but also express the same thought. More specifically, it is claimed that on assumption (2) (i) every single instance of Basic Law V, for example, "$(\dot{\varepsilon}(\varepsilon^2 - 4\varepsilon) = \dot{\alpha}(\alpha\ (\alpha - 4))) = (\forall x(x^2 - 4x = x(x - 4)))$" or "$(\dot{\varepsilon}(-\varepsilon) = \dot{\alpha}(\alpha(\alpha = \alpha))) = (\forall x(-x = (x = (x = x))))$" is epistemically trivial and (ii) Axiom V nonetheless possesses real epistemic value by virtue of its maximal generality and universal validity. In my view, (ii) is flatly at odds with (i). If assumption (2) applies, then the linguistic expression of Axiom V is, from both a semantic and an epistemological point of view, on a par with "$(\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha))$" and "$(\forall x(f(x) = g(x))) = (\forall x(f(x) = g(x)))$". As to the latter equations, it would be pointless to appeal to their validity for every pair of instances of "*f*" and "*g*" in order to demonstrate that they have real epistemic value. Both equations are instances of "$a = a$" and as such lack real epistemic value for the same reason as "$(\dot{\varepsilon}(\varepsilon^2 - 4\varepsilon) = \dot{\alpha}(\alpha\ (\alpha - 4))) = (\dot{\varepsilon}(\varepsilon^2 - 4\varepsilon) = \dot{\alpha}(\alpha\ (\alpha - 4)))$" or "$(\forall x(x^2 - 4x = x(x - 4))) = (\forall x(x^2 - 4x = x(x - 4)))$".

While according to Frege's remark quoted above, the law $a = a$, thanks to its unrestricted generality, might be suited for being selected as an axiom of a logical or mathematical theory T — it applies in fact to every object of an all-embracing domain —once Basic Law V has been transformed into "$(\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha))$" or "$(\forall x(f(x) = g(x))) = (\forall x(f(x) = g(x)))$" by appeal to asssumption (2), the epistemic significance of its original generality — assuming that the truth-value names flanking "=" in Basic Law V, and which embody or represent that generality, express different thoughts — is lost. The residual generality of "$(\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha))$" or "$(\forall x(f(x) = g(x))) = (\forall x(f(x) = g(x)))$" qua instances of "$a = a$" is now subdued to the far greater generality of "$a = a$". "$(\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha))$", for example, is definitely not a primitive truth of logic in Frege's sense; it does not possess utmost generality. Thus, from his point of view, it would not be a candidate for being chosen as a logical axiom. Neither could it be used to introduce value-ranges by stating their identity conditions, nor could it be employed to govern "=" in a logical theory T. In short, the generality inherent in "($\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)$)", which is due to the use of "*f*" and "*g*", is epistemically an idle wheel, and I presume that Frege would have agreed. To reemphasize, "($\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)$)" fares epistemically no better than, for example, "($\dot{\varepsilon}(-\varepsilon) = \dot{\alpha}(\alpha(\alpha = \alpha))) = (\dot{\varepsilon}(-\varepsilon) = \dot{\alpha}(\alpha(\alpha = \alpha))$)". While in Basic Law V the syntactic structure of the truth-value names flanking "=" is considered to be crucial both for its logically relevant generality and its epistemic value — assuming again that these names express different thoughts — the syntactic structure of the truth-value name on both sides of "=" in "($\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)$)" has no impact on the epistemic value of the entire equation. As an instance of "$a = a$", "($\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = (\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)$)" is as good or as bad as "1 = 1".

The upshot is that Frege could not escape the epistemic dilemma under discussion by invoking the feature of maximal generality that in his view belongs to every axiom of his logical system, including Axiom V. If self-evidence is an essential feature of every (logical or non-logical) axiom, and if the requisite self-evidence of Axiom V requires identity of sense of "$\forall x(f(x) = g(x))$" and "($\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)$)", and it surely does, then Axiom V cannot function as a general law that governs value-ranges by determining their identity conditions.

For Frege, one possible moral to be drawn concerning his choice of Axiom V as the key axiom of his logicism might have been this. When before 1893 he embarked on arranging the axiomatic basis of his logical theory, he should have followed his suspicion that Axiom V is not among those axioms that can claim to have a high degree of self-evidence, instead of turning a blind eye to it, hoping that it would remain unchallenged. Frege apparently did not have any misgivings about the other axioms of his logical theory and was convinced that the rules of inference he had laid down were irreproachable, that is, truth-preserving, and that the definitions he had framed were likewise unassailable in the light of the constraints he had imposed on explicit definitions in general. Regarding Axioms I — IV, it seems that in his view self-evidence and non-triviality coexisted peacefully, with the possible exception of the seemingly trivial axiom $|\!-a \rightarrow a$, which is only a special case of $|\!-a \rightarrow (b \rightarrow a)$. Like Axiom V, its nearest kin, namely Axiom VI, appears to be a special case with respect to self-evidence, although it seems that Frege was unaware of this. Recall that Axiom VI is the thought that every object Δ is identical with the value that $\backslash\xi$ has for the value-range $\dot{\varepsilon}(\Delta = \varepsilon)$ as argument. This thought implies of course the coreferentiality of "a" and "$\backslash\dot{\varepsilon}(a = \varepsilon)$" in the formal expression of Axiom VI: $|\!-a = \backslash\dot{\varepsilon}(a = \varepsilon)$. To be sure, Frege did not raise any doubt about the requisite self-evidence of Axiom VI, neither in the first volume of *Grundgesetze* nor in any of his writings after Russell's discovery of the paradox. On the one hand, one could argue as follows: It is true that Frege says only that Axiom VI follows from the *reference* he has assigned to "$\backslash\xi$" by way of elucidation. However, since the formal expression of Axiom VI is an equation of the form "$a = b$", its supposed self-evidence implies that the terms "a" and "$\backslash\dot{\varepsilon}(a = \varepsilon)$" express the same sense. And if they do, then $a = \backslash\dot{\varepsilon}(a = \varepsilon)$ is to be considered epistemically trivial — a fatal consequence. On the other hand, one might wish to argue in this way: It is after all hard to see that the terms "a" and "$\backslash\dot{\varepsilon}(a = \varepsilon)$" should have the same sense, since "a" is here only a constituent of "$\backslash\dot{\varepsilon}(a = \varepsilon)$". But if there is no sameness of sense, how can Axiom VI be self-evident? On the face of it, this would defy explanation. Which assessment should be regarded as the right one may be controversial.[66] But let us return to Axiom V. From Frege's remarks in the aftermath of Russell's paradox it seems pretty clear that dispensing with Axiom V or a modified version of it would not have been an option for him. The simple reason is that he considered this axiom to be the irreplaceable linchpin of his logicism. Thus, abandoning Axiom V or a revised version of it would have been tantamount to abandoning logicism altogether.

At the outset of section 6, I have argued that regarding Axiom V Frege cannot have it both ways, whichever way you look at it: self-evidence and real epistemic value. A magic key to solve the puzzle is not to hand, nor can Frege hope for help from *deus ex machina*. But could he nevertheless have traced out a viable option to escape this epistemic dilemma? Following a piece of advice by Cicero (*De Officiis,* 3.3), I suggest: Of two evils choose the lesser ("De duobus malis minus est eligendum"). And the lesser evil for Frege would have been dropping or weakening the condition of self-evidence while retaining the requirement of genuine knowledge or real epistemic value. More specifically, I think that in principle he could perhaps have replaced the notion of self-evidence as applied to axioms by a weaker epistemic notion that (a) allows for a lower degree of evidence than that required by him for axioms, (b) does not rely on or imply the synonymy of the two sides of the contextual stipulation or Basic Law V and, hence, (c) helps to avoid the disastrous consequence of degenerating Basic Law V to an epistemically trivial instance of "$a = a$" which could not do any work in the logical construction of arithmetic. I suggest we tentatively consider, for example, the notion of intrinsic plausibility to play such a role.

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[66] Thanks to Philip Ebert for drawing my attention to Basic Law VI in the relevant context.

For someone familiar with set theory all the axioms of ZFC probably have what Charles Parsons (cf. *Parsons 2008*, pp. 319 ff., 338) calls *intrinsic plausibility*. According to his use of this phrase, *obviousness* or *self-evidence* may generally imply a higher degree of evidence than *intrinsic plausibility*. Intuitively speaking, I can see the difference that Parsons has in mind regarding the notions of self-evidence and intrinsic plausibility. Anyway, adopting the sense that Frege attaches to the phrase "(*unmittelbar*) *einleuchtend*" one may wish to say: If any of the axioms of ZFC can claim to be self-evident, then it is the axiom of extensionality. This axiom expresses the fundamental idea of a set as opposed to an intensional entity such as a property: "Every set is determined by its elements" (*Zermelo 1908*, p. 201). The axiom of pairing and the axiom declaring the existence of a null set (termed by Zermelo "axiom of elementary sets") presumably come close to being self-evident. Yet it seems that things are different for the axioms of infinity, replacement, power set, and choice. Although I believe that on intuitive grounds we would deny that they are self-evident, we may still want to regard them as intrinsically plausible (see in this connection also the brief discussion in *Parsons 2008*, pp. 338 ff.).[67] However, in the higher region of set theory involving large cardinals, whose existence is incompatible with V = L, the role of intrinsic plausibility is much diminished. Here I agree with Parsons (p. 341). As far as the Peano axioms are concerned, we would probably not hesitate to characterize them as intrinsically plausible. But, as Parsons observes (p. 332), these axioms are not intrinsically plausible by themselves. Their intrinsic plausibility is buttressed by the network of connections in which they stand.[68]

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[67] As far as I can see, the question concerning the self-evidence of the axioms of ZF or ZFC is discussed neither in *Zermelo 1908*, nor in *Fraenkel 1921, 1922, 1922a*, *1922b, 1923*, *1924*, *1925*, nor in *Skolem 1922*, nor does von Neumann raise the issue of self-evidence for his new axiom system in his article 'Eine Axiomatisierung der Mengenlehre' (1925). Thus, it seems that the issue of self-evidence did not loom large for them when they worked out an axiom system or modified or criticized an existing one or reflected on its admissibility, strength or justification.

[68] Regarding Frege's axioms in *Grundgesetze*, we might alternatively introduce the notion of logical transparency of a proposition which, like the notion of intrinsic plausibility, is supposed to allow for a lower degree of evidence than Frege's notion of self-evidence. Generally speaking, the logical transparency is assumed to be that property of a proposition which gives rise to or motivates our non-inferential acknowledgement of its truth and, moreover, induces us to recognize it as a truth of logic. As in the case of self-evidence and intrinsic plausibility, it is hereby presupposed that the content or meaning of the sentence under consideration is fully grasped by the judging person. But what does a person's full grasp of, for example, Hume's Principle or Basic Law V amount to or involve? A clear-cut answer is probably not at hand; see in this respect the brief discussion in *Schirn 2014a*.

Clearly, the notions of intrinsic plausibility, logical transparency and their kin, which are supposed to require a lower degree of evidence than Frege's notion of self-evidence, are open to controversy. They suffer from considerable vagueness as does Frege's notion of self-evidence. But note that here I only want to make the tentative proposal that in the face of the epistemic predicament arising from Axiom V Frege might have seen a chance to wriggle out of it by replacing the strong notion of self-evidence with the weaker notion of intrinsic plausibility that, unlike the former notion, allows for a difference of sense of (a) and (b) in Basic Law V.

In summary, we can say that in Frege's view stipulating the coreferentiality of the generality of an equation between function-values and the corresponding value-range equation suffices for fixing partially the reference of the value-range operator. And, as we have seen, the unfinished business left by the contextual stipulation in §3 is supposed to be completed by further stipulations relating primarily and essentially to the *references* of certain primitive and non-primitive concept-script names. Self-evidence of the contextual stipulation, which in my view would have to rest on the sameness of the senses of its two sides, is not required for the purpose of endowing each canonical value-range term with a unique reference by means of this stipulation and the additional stipulations in §§10-12. Nor is synonymy of the two sides of Basic Law V a necessary condition for securing its requisite logical nature. Recall my claim that it is very unlikely that Frege based its logical status on the assumption of sense identity. And we know of course that in pursuit of his logicism he acknowledged the existence of logically true identity sentences of the form "*a* = *b*", where "*a*" and "*b*" express different senses. Yet although I have argued that for some reason(s) Frege refrained from asserting synonymy of the truth-value names (a) and (b) flanking the main identity sign in Basic Law V, he most likely saw a close contentual connection between (a) and (b) or as we might also say: an intimate logical relation between the thoughts expressed by (a) and (b).[69] If so, then it is precisely this connection or relation that Frege should have explained in a plausible way in order to have at least one strong argument for the required purely logical nature of Basic Law V.[70] Finally, for Basic Law V in order to be accepted as the means that affords us the right cognitive access to value-ranges via a grasp of their identity conditions (supported by the stipulations made in §§10-12), it is again not mandatory to insist on self-

[69] By contrast and almost trivially, there is no such contentual connection between, for example, the truth-value names flanking the main identity sign in the true equation "$(\grave{\varepsilon}(-\varepsilon) = \dot{\alpha}(\alpha(\alpha = \alpha))) = (2 + 2 = 4)$."

[70] Recall also my remark in section 2 that by appealing to the *internal* structure of Basic Law V Frege might have seen a chance to consider this law to be true by virtue of its form.

evidence and, hence, on the synonymy of (a) and (b). The weaker notion of intrinsic plausibility, if accepted as basically viable, could after all do a better job since it seems to allow for a difference of sense of (a) and (b) in Basic Law V. And this supposed semantic difference not only saves Basic Law V from epistemic triviality, but I think that it also renders more intelligible the act of apprehending value-ranges via abstraction[71], which Frege considered to be the key to his entire logicist project.

7. Frege's reactions to Russell's paradox in the period 1902-1906

In this concluding section, I take a critical look at Frege's reactions to Russell's paradox in the period 1902-1906. Moreover, I speculate about Frege's position regarding a logical foundation of arithmetic had he kept abreast of the development of axiomatized set theory.

Russell's paradox hit Frege to the core. Needless to say, he could not pull a solution out of a hat. His reactions to the paradox in the period 1902-1906 rather suggest that he was at his wit's end regarding the projected logical foundation of arithmetic. His posthumously published essay 'Über Schoenflies: Die logischen Paradoxien der Mengenlehre' (1906) begins with a number of surprising notes which he presumably jotted down before embarking on writing the piece (*Frege 1969*, p. 191):

> Russell's contradiction cannot be eliminated in Schoenflies's manner. Concepts that coincide in extension, although this extension falls under the one, but not under the other. Remedy from extensions of second-level concepts [is] impossible. Set theory shattered. My concept-script in the main independent of that.

Frege is possibly not quite right here as far as his assessment of the fate of set theory in the light of the logical paradoxes is concerned. It is true that Russell's and Zermelo's discovery of the paradoxes lurking in naïve set theory with its unrestricted comprehension principle[72] shook the logicist foundations of cardinal arithmetic and analysis, but it is likewise true that Frege's theory of value-ranges was only a special variant of such a theory. Thus, how could Frege intelligibly claim that his concept-script was in the main independent of the breakdown of (naïve) set theory? Or did he mean that the second-order fragment of his overall logical

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[71] The step of abstraction in Basic Law V, that is, the recognition of something common to two monadic first-level functions *f* and *g* (cf. *Frege 1903*, §146), proceeds from right to left. It is formally represented by Va: $(\forall x(f(x) = g(x))) \rightarrow (\dot{\varepsilon} f(\varepsilon) = \dot{\alpha} g(\alpha))$. And recall that Va can claim to be regarded as a logical truth.

[72] Frege's generalization of the comprehension principle is Theorem 1 of *Grundgesetze*: $f(a) = a \in \dot{\varepsilon} f(\varepsilon)$. I have replaced his symbol for the membership function with "∈".

theory was independent of that breakdown? If so, this would confirm a hunch I voiced at the outset of this paper, namely that in Frege's view Russell's paradox did not affect second-order logic.

Later in the essay on Schoenflies, Frege tends to take a rather irresolute attitude when assessing the impact that Russell's paradox had on Axiom V. On the one hand, he admits that Axiom V is not as evident as one would wish for a law of logic and that possible previous doubts concerning that axiom were reinforced by Russell's paradox. On the other hand, he states that in the case of the transformation of the generality of a function-value equality into a value-range identity we must assume an unprovable law (cf. also *Frege 1893*, §9; *Frege 1903*, §147; *Frege 1967*, p. 130).[73] How can he pretend that nothing really devastating to Axiom V had been revealed by Russell's paradox so that one might almost proceed to the order of the day ("Yet let us set aside these doubts about Axiom V for the moment", *Frege 1969*, p. 198)? In particular, how can Frege insinuate that Axiom V might after all survive as an *unprovable law of logic*? At that time, four years after Russell's startling discovery, it must have been clear to him that Axiom V was irrevocably lost. It was only near the end of his life, after a period of increasing doubt about the viability of his logicist project, that Frege abandoned it in explicit form, being convinced that it was a total failure. He then turned, though in a fragmentary fashion, to a geometrical foundation of arithmetic, thus giving up another conviction he had forcefully defended from the beginning of his career, namely that the principles of arithmetic and those of geometry are to be justified in fundamentally different ways (cf. *Frege 1967*. p. 50).[74]

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[73] Prior to this claim, Frege makes an interesting remark on the apparent unavoidability of the transition from right to left in Basic Law V; he confines himself to mentioning the case of concepts and their extensions which can of course be generalized to the case of functions and their value-ranges: The fact that the properties of reflexivity, symmetry and transitivity of identity have their analogues for the case of the second-level relation of coextensiveness between first-level functions "compels us almost ineluctably to transform a sentence" in which coextensiveness is asserted of functions into a sentence expressing an equality (cf. *Frege 1969*, pp. 197 f.).

[74] As far as Frege's late idea of providing a geometrical foundation of arithmetic is concerned, it was, to my mind, not remotely a new awakening but only a desperate move, indeed a non-starter. And I trust that he had at least an inkling that proposing such a sea change in his philosophy of arithmetic and his foundational outlook in general did not carry an awful lot of conviction. I fail to see, by any stretch of the imagination, that the geometrical source of knowledge qua spatial intuition, which — as Frege stresses more than once — is far more restricted in scope than the logical source of knowledge, could persuasively account for the distinguishing marks setting arithmetic apart from intuition-based geometry.

To be sure, around 1906, when Frege wrote the piece on Schoenflies, set theory in general was by no means in ruins. On the contrary, soon after Russell's and Zermelo's discovery of the paradox, *axiomatized* set theory was already in *statu nascendi*,[75] and soon in Zermelo's groundbreaking essay 'Untersuchungen über die Grundlagen der Mengenlehre I' (1908) the first axiom system for set theory saw the light of day. The emergence of the Burali-Forti and Russell paradoxes was ruled out in Zermelo's system, because both the set of all ordinals and the set of all sets that do not contain themselves as elements were banished from that system. Thanks to the pioneering work of Zermelo and the later developments of set theory initiated by Fraenkel (1921, 1922, 1922a, 1922b, 1923, 1924, 1925), Skolem (1922),[76] and von Neumann (1925), axiomatized set theory began to flourish and quickly bore fruit in logic and the foundations of mathematics.

Had Frege kept abreast of the development of axiomatized set theory — which he apparently failed to do for reasons we need not go into here — he might have seen an opportunity coming up: to salvage the fundamental idea of logicism. I assume though that the prospects for vindicating the viability of logicism in the light of the results achieved by the avant-garde of axiomatized set theory would have been poor from Frege's standpoint, unless

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[75] In *Zermelo 1904*, Zermelo published his first proof of the well-ordering theorem. I presume that already at that time he was reflecting intensively on an appropriate axiomatic basis for set theory, one that would avoid the pitfalls of naïve set theory. In the first part of his 1908a paper, he presented a new proof for the well-ordering theorem. Like the original proof, the new proof relied essentially on a key axiom of his system in *Zermelo 1908*, namely the axiom of choice that he had formulated for the first time in *Zermelo 1904*. In the second part of his 1908a paper, Zermelo discusses the massive objections that were raised to his first proof. In *Zermelo 1909* and *1909a*, he deals with arithmetic in set theory.

[76] In spite of the criticisms that Skolem levelled against Zermelo's axiomatization — for example, (a) that the vagueness looming in the concept of a *definite proposition* must be eschewed, (b) that Zermelo's set theory is limited in the sense that it does not ensure the existence of some "large" sets, (c) that the discrepancy between an intuitive set-theoretic concept and its formal counterpart involves the "relativity" of set-theoretic notions, and (d) that, due to their non-categoricity, Zermelo's axioms probably do not provide the appropriate means of deciding all cardinality problems — it would be thoroughly wrong to say that Skolem was opposed to (the rise of) axiomatized set theory. He just felt competent enough to point out a number of deficiencies in Zermelo's axiomatization (he actually made eight points) in order to prevent misjudgement among mathematicians. Furthermore, he tried to remedy some of them. In doing this, he undoubtedly contributed to the future advances in axiomatized set theory, although he may not have regarded it as the ideal foundation for mathematics. The concluding statement in his 1922 paper (p. 301) speaks for itself: "But in recent times I have seen to my surprise that so many mathematicians think that these axioms of set theory provide the ideal foundation for mathematics, therefore it seemed to me that the time had come to publish a critique."

he had been prepared to make significant changes in his conception of the notions of axiom, of consistency and of logic in general. To mention the key condition for his potential acceptance of axiomatized set theory as a foundation of arithmetic: Only if he believed that Zermelo and his fellow protagonists could justifiably establish set theory as a proper part of logic, might he have thought that logicism was not inevitably doomed to failure when arithmetic was to be grounded on axiomatized set theory.

Russell and Whitehead attempted to defend logicism by constructing a logical theory based on a ramified theory of types together with the axioms of reducibility, of infinity and of choice. Due to the evidently non-logical character at least of the former two axioms their enterprise could not be taken to vindicate the claims of logicism; rather it seems to have fostered the decline of logicism. We do not know of any reaction of Frege's concerning the status of Russell and Whitehead's axioms. Nonetheless, I guess that Frege would have rejected out of hand both the axiom of reducibility and the axiom of infinity by arguing that they are neither self-evident nor purely logical. Admittedly, at least at the time when he wrote the first volume of *Grundgesetze* the boundary of his notion of (self-) evidence seems to have been a little fuzzy. Had Frege taken pains to delimit the scope of its application more accurately, he might have refrained from including the mutual transformation of the generality of a function-value equality into a value-range identity in the set of the axioms of his logical theory. Yet, as I said earlier, from the point of view he held in the period 1893-1903, this would have been tantamount to throwing overboard the logicist programme altogether.

Finally, let me mention in this connection that Frege's papers 'On the Foundations of Geometry' (1903 and 1906) suggest that even in the aftermath of Russell's and Zermelo's discovery Frege was unwilling to revise his Euclidean conception of axioms and his related approach to the need and role of carrying out consistency proofs for contentually interpreted mathematical theories. Frege objected to the consistency proof, carried out by Hilbert in *Hilbert 1899* for an axiom system of Euclidean geometry, that consistency follows immediately from the truth of the axioms, assuming that the axioms were genuine and not Hilbertian axioms qua implicit definitions, which Frege stigmatized as pseudo axioms. Since he regarded genuine axioms as necessarily true, at least those of a theory that could lay claim to being purely logical, he did not think that they could contradict each other.[77] All this has an air of tragic irony when we think of the definite failure of Frege's logicist project, standing

[77] Concerning Frege's view on consistency see *Dummett 1976*, *Blanchette 1996, 2007, 2012* and *Schirn 2010*.

out in the philosophy of mathematics of his time due to its crystal clarity, unrivalled depth and intellectual honesty.[78]

References

Blanchette, P.: 1996, 'Frege and Hilbert on Consistency', *Journal of Philosophy* 93, 317-336.

Blanchette, P.: 2007, 'Frege on Consistency and Conceptual Analysis', *Philosophia Mathematica* 15, 321-346.

Blanchette, P.: 2012, *Frege's Conception of Logic*, Oxford University Press, New York.

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Boolos, G.: 1987, 'The Consistency of Frege's Foundations of Arithmetic', in J.J. Thomson (ed.), *On Being and Saying. Essays for Richard Cartwright*, The MIT Press, Cambridge, Massachusetts, 3-20.

Boolos, G.: 1993, 'Whence the Contradiction', *Proceedings of the Aristotelian Society Supplementary Volume* LXVII, 213-233; reprinted in Schirn 1996, 234-252.

Boolos, G.: 1997, 'Is Hume's Principle Analytic?', in Heck 1997, 245-261.

Bueno, O.: 2010, 'A Defense of Second-Order Logic', in *Axiomathes* 20, 365-383.

Burge, T.: 1984, 'Frege on Extensions of Concepts, From 1884 to 1903', *The Philosophical Review* 93 (1984), 3-34.

Burge, T.: 1998, 'Frege on Knowing the Foundation', *Mind* 107, 305-347; reprinted in Burge, *Truth, Thought, Reason. Essays on Frege*, Oxford University Press, Oxford, 2005, 317-355.

Dummett, M.: 1973, *Frege. Philosophy of language*, Duckworth, London.

Dummett, M.: 1976, 'Frege on the Consistency of Mathematical Theories', in Schirn 1976, vol. I, *Logik und Philosophie der Mathematik — Logic and Philosophy of Mathematics*, 229-242.

Dummett, M.: (1991), *Frege. Philosophy of Mathematics*, Duckworth, London.

Dummett, M.: 1994, 'Reply to Boolos', *Proceedings of the Aristotelian Society* XCIV, 243-249; reprinted in Schirn 1996, 253-260.

Dummett, M.: 1998, 'Neo-Fregeans. In Bad Company', in Schirn 1998, 369-387.

Ebert, P.: 2008, 'A Puzzle About Ontological Commitments', *Philosophia Mathematica* 16, 209-226.

Ebert, P.: 2016, 'Frege on Sense-Identity, Basic Law V, and Analysis', forthcoming.

Fine K.: 1998, 'The Limits of Abstraction', in Schirn 1998, 503-629.

Fraenkel, A. A.: 1921, 'Über die Zermelosche Begründung der Mengenlehre', *Jahresbericht der Deutschen Mathematiker-Vereinigung* 30, 97-98.

Fraenkel, A. A.: 1922, 'Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre', *Mathematische Annalen* 86, 230-237.

Fraenkel, A. A.: 1922a, 'Der Begriff "definit" und die Unabhängigkeit des Auswahlaxioms', *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse*, 253-257.

Fraenkel, A. A.: 1922b, 'Zu den Grundlagen der Mengenlehre', *Jahresbericht der Deutschen Mathematiker-Vereinigung* 31, 101-102.

Fraenkel, A. A.: 1923, *Einleitung in die Mengenlehre*, Springer, Berlin.

Fraenkel, A. A.: 1924, 'Die neueren Ideen zur Grundlegung der Analysis und der Mengenlehre', *Jahresbericht der Deutschen Mathematiker-Vereinigung* 33, 97-103.

Fraenkel, A. A.: 1925, 'Untersuchungen über die Grundlagen der Mengenlehre', *Mathematische Zeitschrift 22,* 250-273.

Frege, G.: 1879, *Begriffsschrift. Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens,* L. Nebert, Halle a. S.

Frege, G.: 1884, *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl,* W. Koebner, Breslau.

Frege, G.: 1893, *Grundgesetze der Arithmetik. Begriffsschriftlich abgeleitet,* vol. I, H. Pohle, Jena.

Frege, G.: 1903, *Grundgesetze der Arithmetik. Begriffsschriftlich abgeleitet,* vol. II, H. Pohle, Jena.

Frege, G.: 1967, *Kleine Schriften,* ed. I. Angelelli, G. Olms, Hildesheim.

Frege, G.: 1969, *Nachgelassene Schriften,* eds. H. Hermes, F. Kambartel and F. Kaulbach, F. Meiner, Hamburg.

Frege, G.: 1976, *Wissenschaftlicher Briefwechsel,* eds. G. Gabriel, H. Hermes, F. Kambartel, C. Thiel, A. Veraart, F. Meiner, Hamburg.

Heck, R. G.: 1995, 'Frege's Principle', in *From Dedekind to Gödel*, ed. J. Hintikka, Kluwer, Dordrecht.

Heck, R. G. (ed.): 1997, *Language, Thought, and Logic. Essays in Honour of Michael Dummett*, Clarendon Press, Oxford.

Heck, R. G.: 1997a, '*Grundgesetze der Arithmetik* I §§29-32', *Notre Dame Journal of Formal Logic* 38, 437-474.

Heck, R. G.: 1999, '*Grundgesetze der Arithmetik* I §10', *Philosophia Mathematica* 7, 258-292.

Heck, R. G.: 2005, 'Julius Caesar and Basic Law V', *Dialectica* 59, 161-178.

Heck, R. G.: 2011, *Frege's Theorem*, Oxford University Press, Oxford.

Hilbert, D.: 1899, *Grundlagen der Geometrie*, Teubner, Leipzig; seventh revised and enlarged edition 1930.

Von Kutschera F.: 1966, 'Freges Begründung der Analysis', Ar*chiv für mathematische Logik und Grundlagenforschung* 9, 102-111; reprinted in Schirn 1976, vol. I, 301-312.

Leibniz, G. W.: 1875-1890, *Die philosophischen Schriften von G.W. Leibniz,* ed. C. J. Gerhardt, Berlin; reprint G. Olms, Hildesheim 1960-1961.

Leibniz, G. W.: 1903, *Opuscules et fragments inédits de Leibniz*, ed. L. Couturat, Paris; reprint G. Olms, Hildesheim 1966.

Linnebo, O.: 2004, 'Frege's Proof of Referentiality', *Notre Dame Journal of Formal Logic* 45, 73-98.

Linnebo, O. (ed.): 2009, *Synthese* 170, special issue on the bad company objection, 321-482.

MacFarlane, J.: 2002, 'Frege, Kant, and the Logic in Logicism', *The Philosophical Review* 11, 25-63.

May, R. and Wehmeier, K. F.: 2017, 'The Proof of Hume's Principle', in this volume.

Parsons, C.: 2008, *Mathematical Thought and Its Objects*, Cambridge University Press, Cambridge.

Pedriali, W.: 2017, 'Epistemology of the Basic Laws', in this volume.

Potter, M.: 2000, *Reason's Nearest Kin. Philosophies of Arithmetic from Kant to Carnap*, Oxford University Press, Oxford.

Ruffino, M.: 2003, 'Why Frege Would Not Be a Neo-Fregean', *Mind* 112, 51-78.

Russell, B.: 1903, *The Principles of Mathematics*, Cambridge; second edition, New York.

Schirn, M. (ed.): 1976, *Studien zu Frege — Studies on Frege* I - III, Stuttgart-Bad Canstatt.

Schirn, M.: 1983, 'Begriff und Begriffsumfang. Zu Freges Anzahldefinition in den *Grundlagen der Arithmetik'*, *History and Philosophy of Logic* 4, 117-143.

Schirn, M: 1988, Review Essay on G. Frege, *Die Grundlagen der Arithmetik*, centenary edition, ed. C. Thiel, Felix Meiner, Hamburg 1986, *The Journal of Symbolic Logic* 83, 993-999.

Schirn, M.: 1989, 'Frege on the Purpose and Fruitfulness of Definitions', *Logique et Analyse* 125-126, 61-80.

Schirn, M.: 1994, 'Frege y los nombres de cursos de valores', *Theoria* 9, 109-133.

Schirn, M. (ed.): 1996, *Frege: Importance and Legacy*, Walter de Gruyter, Berlin, New York.

Schirn, M.: 1996a, 'On Frege's Introduction of Cardinal Numbers as Logical Objects', in Schirn 1996, 114-173.

Schirn, M. (ed): 1998, *The Philosophy of Mathematics Today*, Oxford University Press, Oxford.

Schirn, M.: 2003, 'Fregean Abstraction, Referential Indeterminacy and the Logical Foundations of Arithmetic', *Erkenntnis* 59, 203-232.

Schirn, M.: 2006, 'Hume's Principle and Axiom V Reconsidered: Critical Reflections on Frege and his Interpreters', *Synthese* 148, 171-227.

Schirn, M.: 2006a, 'Concepts, Extensions, and Frege's Logicist Project', *Mind* 115, 983-1005.

Schirn, M.: 2009, 'La definición logicista de número', *Analítica* 3, 7-46.

Schirn, M.: 2010, 'Consistency, Models, and Soundness', *Axiomathes* 20, 153-207.

Schirn, M.: 2010a, 'On Translating Frege's *Die Grundlagen der Arithmetik*', *History and Philosophy of Logic* 31, 47-72.

Schirn, M.: 2013, 'Frege's Approach to the Foundations of Analysis (1873-1903)', *History and Philosophy of Logic* 34, 266-292.

Schirn, M.: 2014, 'Frege on Quantities and Real Numbers in Consideration of the Theories of Cantor, Russell and Others', in G. Link (ed.), *Formalism and Beyond. On the Nature of Mathematical Discourse,* Walter de Gruyter, Boston and Berlin, 25-95.

Schirn, M.: 2014a, 'Frege's Logicism and the Neo-Fregean Project', *Axiomathes* 24, 207-243.

Schirn, M.: 2016, 'On the Nature, Status, and Proof of Hume's Principle in Frege's Logicist Project', in V. S. Costreie (ed.), *Early Analytic Philosophy. New Perspectives on the Tradition*, Springer, 49-96.

Schirn, M.: 2016a, Review of Gottlob Frege, *Basic Laws of Arithmetic. Derived Using Concept-Script*. Vols. I & II, translated and edited by P. A. Ebert and M. Rossberg with C. Wright, Oxford University Press, Oxford 2013, *The Philosophical Quarterly* 67.

Schirn, M.: 2017, 'The Semantics of Value-Range Names and Frege's Proof of Referentiality', forthcoming.

Schirn, M.: 2017a, *Funktion, Gegenstand, Bedeutung. Freges Philosophie und Logik im Kontext*, forthcoming, mentis, Münster.

Schirn, M.: 2017b, *Frege on the Foundations of Mathematics*, *Synthese Library*, *Studies in Epistemology, Logic, Methodology, and Philosophy of Science*, editor-in-chief: O. Bueno, forthcoming, Springer.

Simons, P.: 1987, 'Frege's Theory of Real Numbers', *History and Philosophy of Logic* 8, 25-44.

Simons, P.: 1992, 'Why Is There So Little Sense in *Grundgesetze*?', *Mind* 101, 753-766.

Skolem, T.: 1922, 'Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre', *Mathematikerkongressen i Helsingfors, 4-7 July 1922*; 'Den Fente skandinaviska mathematikerkongressen', 217-232; English translation in van Heijenoort 1967, 290-301.

Sluga, H.: 1980, *Gottlob Frege*, Routledge & Kegan Paul, London.

Sluga, H.: 1986, 'Semantic Content and Cognitive Sense', in L. Haaparanta and J. Hintikka (eds.), *Frege Synthesized*, Kluwer, Dordrecht, 47-64.

van Heijenoort, J. (ed.): 1967, *From Frege to Gödel. A Soure Book in Mathematical Logic*, 1879-1931, Harvard University Presss, Cambridge, Mass.

von Neumann, J.: 1925, 'Eine Axiomatisierung der Mengenlehre', *Journal für die reine und angewandte Mathematik* 154, 219-240.

Wehmeier, K. F. and Schroeder-Heister, P.: 2005, 'Frege's Permutation Argument Revisited', *Synthese* 147, 43-61.

Wright, C.: 1998, 'On the Harmless Impredicativity of $N^=$ ("Hume's Principle")', in Schirn 1998, 339-368.

Wright, C.: 1998a, 'Response to Dummett', in Schirn 1998, 389-405.

Zermelo, E.: 1904, 'Beweis daß jede Menge wohlgeordnet werden kann', *Mathematische Annalen* 59, 514-516.

Zermelo, E.: 1908, 'Untersuchungen über die Grundlagen der Mengenlehre I', *Mathematische Annalen* 65, 261-281; English translation in van Heijenoort 1967, 199-215.

Zermelo, E.: 1908a, 'Neuer Beweis für die Möglichkeit einer Wohlordnung', *Mathematische Annalen* 65, 107-128.

Zermelo, E.: 1909, 'Über die Grundlagen der Arithmetik', *Atti del IV Congresso Internazionale dei Matematici*, Roma, 2, 8-11.

Zermelo, E.: 1909a, 'Sur les ensembles finis et le principe de l'induction complète', *Acta Mathematica* 32, 185-193.