Self-referentiality in the framework of justification logics

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Outline

- Realization in Justification Logic
- Self-referentiality
- Properties of non-self-referential fragments

- Realization in Justification Logic

Justification logics JL

- Explicit versions of modal logics ML.
  - $\Box\phi$ v.s. $t{:}\phi$,
  - t explains contents implicitly indicated by $\Box$.
- Language: propositional, extended by $t{:}\phi$.
  - t is a term (inductively defined, sensitive to logics),
  - ϕ is a formula in **this** language (where terms may occur in).
- The family of JL: >30 members, serving as explicit versions to many well-known ML's.
  - We will focus on the following five pairs:

ML	K	D	T	K4	S4
JL	J	JD	JT	J4	LP

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The Logic of Proofs LP as an example

- By Artemov in 1995.
- $\phi := \bot \mid p \mid \phi \to \phi \mid t{:}\phi$,
  $t := c \mid x \mid t \cdot t \mid t + t \mid !t$.
- Axiom schemes:
  - Classical propositional axioms,
  - $t{:}\phi \to \phi$,
  - $t_1{:}(\phi \to \psi) \to (t_2{:}\phi \to t_1 \cdot t_2{:}\psi)$,
  - $t{:}\phi \to !t{:}t{:}\phi$,
  - $t_1{:}\phi \to t_1 + t_2{:}\phi$ and $t_2{:}\phi \to t_1 + t_2{:}\phi$.
- Rules schemes:
  - $\alpha \to \beta, \alpha \vdash \beta$,
  - $\vdash c : A$, where c is a constant, and A is an axiom.
- Explicit version of modal logic S4.
- Formally, the implicit/explicit correspondence is called realization.

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Realization

- Realizer
  - A mapping: the language of ML $\rightsquigarrow$ that of **a** JL;
  - Assigns a term to each $\Box$-occurrence in the input formula.
- Realization
  - Given realizer $(\cdot)^r$ and modal formula ϕ, the image ϕ^r is a potential realization;
  - ϕ^r is a realization if further $\mathsf{JL} \vdash \phi^r$.

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Realization (continued)

- Realization theorem (Artemov 1995 & Brezhnev 2000)
  - For any modal formula ϕ:
    - Let $\mathsf{X} \in \{\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{S4}\}$,
    - and $\mathsf{Y} \in \{\mathsf{J}, \mathsf{JD}, \mathsf{JT}, \mathsf{J4}, \mathsf{LP}\}$, resp.,
  - Then what follows are equivalent:
    - $\mathsf{X} \vdash \phi$;
    - $\mathsf{Y} \vdash \phi^r$ for some realizer $(\cdot)^r$.

- Self-referentiality

Self-referential JL-formulas

- (recalled) Justification language (LP as an example)
  - Formula $\phi := \bot \mid p \mid \phi \to \phi \mid t : \phi$;
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- self-referential formulas like $t : \phi(t)$
  - even $c : A(c)$ is possible.

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Constant specification $\mathcal{CS}$

- Definition (take LP as our example):
  - A set of formulas of the form $c : A$.
- Link axioms with constants that present them in terms.
- JL($\mathcal{CS}$) is the fragment of JL where rule scheme *AN* can only put formulas from $\mathcal{CS}$.
  - e.g., JL($\varnothing$) is the fragment of JL without *AN*.

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Self-referentiality of $\mathcal{CS}$

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- $\mathcal{CS}$ is (directly) self-referential, if for some c and A

$$c\!:\!A(c) \in \mathcal{CS}.$$

- Let $\mathcal{CS}^* := \{c\!:\!A \mid c \text{ does not occur in } A\}$;
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  - Thus, JL($\mathcal{CS}^*$) is the fragment of JL where *AN* can only introduce non-self-referential formulas.

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ML^{NR}: non-self-referential realizable fragment of ML

- Definition:
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  - $\mathrm{X}^{NR} := \{\mathrm{X} \vdash \phi \mid \mathrm{Y}(\mathcal{CS}^*) \vdash \phi^r$ for some realizer $(\cdot)^r\}$.
  - A model theorem is non-self-referential if being in ML^{NR}, and self-referential otherwise.
- Self-referential modal-theorems exist. (Kuznets 2006 & 2008):
  - $\mathrm{K}^{NR} = \mathrm{K}$
  - $\mathrm{D}^{NR} = \mathrm{D}$
  - $\Diamond(p \to \Box p) \in \mathrm{T} \setminus \mathrm{T}^{NR}$
  - $\Box\neg(p \to \Box p) \to \Box\bot \in \mathrm{K4} \setminus \mathrm{K4}^{NR}$
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Realizing intuitionistic propositional logic IPC via S4

- The initial motivation of Artemov's LP;
- The Gödel–Artemov formalization of BHK semantics;
- Gödel's modal embedding $(\cdot)^{\triangle}$ is a mapping from propositional language to propositional modal language that satisfies:

$$\begin{cases} p^{\triangle} = \Box p \\ \bot^{\triangle} = \Box\bot \\ (\phi \oplus \psi)^{\triangle} = \Box(\phi^{\triangle} \oplus \psi^{\triangle}) \text{ for } \oplus \in \{\wedge, \vee, \to\}. \end{cases}$$

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Basic embeddings

- An extension of Gödel's modal embedding.
- A potential embedding $((\cdot)^{\times})$ is basic if (let $\odot \in \{\wedge, \vee\}$):

$$\begin{cases} \phi^{\times} = \phi^{\times}_{+} \\ p^{\times}_{+} = \Box^{h_{+}} p \qquad p^{\times}_{-} = \Box^{h_{-}} p \qquad \text{similar for } \bot \\ (\phi \odot \psi)^{\times}_{+} = \Box^{j_{\odot+}}(\Box^{k_{\odot+}} \phi^{\times}_{+} \odot \Box^{l_{\odot+}} \psi^{\times}_{+}) \\ (\phi \odot \psi)^{\times}_{-} = \Box^{j_{\odot-}}(\Box^{k_{\odot-}} \phi^{\times}_{-} \odot \Box^{l_{\odot-}} \psi^{\times}_{-}) \\ (\phi \to \psi)^{\times}_{+} = \Box^{j_{\to+}}(\Box^{k_{\to+}} \phi^{\times}_{-} \to \Box^{l_{\to+}} \psi^{\times}_{+}) \\ (\phi \to \psi)^{\times}_{-} = \Box^{j_{\to-}}(\Box^{k_{\to-}} \phi^{\times}_{+} \to \Box^{l_{\to-}} \psi^{\times}_{-}) \end{cases}$$

- A basic embedding is a potential one that satisfies: IPC $\vdash \phi$ iff S4 $\vdash \phi^{\times}$.
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- Definition:
  - $\mathsf{IPC}^{NR(\times)} := \{\mathsf{IPC} \vdash \phi \mid \phi^{\times} \in \mathsf{S4}^{NR}\}$;
  - $\mathsf{IPC}^{NR} := \bigcup_{\times} \mathsf{IPC}^{NR(\times)}$;
  - An intuitionistic theorem is non-self-referential if being in IPC^{NR}, and self-referential otherwise;
  - $\mathsf{IPC}_{\to}^{NR(\times)}$ and $\mathsf{IPC}_{\to}^{NR}$ are similarly defined based on $\mathsf{IPC}_{\to}$.
- Self-referential IPC-theorem exists. (Yu 2014):
  - $\{\neg\neg\alpha \mid \alpha \in \mathsf{CPC} \setminus \mathsf{IPC}\} \subseteq \mathsf{IPC} \setminus \mathsf{IPC}^{NR}$
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- Properties of non-self-referential realizable fragments

Prehistoric-cycle-free provable fragment

- For each logic mentioned above,
  the *CF* (prehistoric-cycle-free provable) fragment
  is a subset of
  the *NR* (non-self-referential realizable) fragment;
  - The best known approximation;
  - Decidable, wieldy for simple formulas.

The underline calculus **G3**[**st4**]

$Ax.\quad \dfrac{}{p, \Gamma \Rightarrow \Delta, p}$ $\qquad L\bot.\quad \dfrac{}{\bot, \Gamma \Rightarrow \Delta}$

$L\rightarrow.\quad \dfrac{\Gamma \Rightarrow \Delta, \phi \quad \psi, \Gamma \Rightarrow \Delta}{\phi \rightarrow \psi, \Gamma \Rightarrow \Delta}$ $\qquad R\rightarrow.\quad \dfrac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \rightarrow \psi}$

$L\Box.\quad \dfrac{\theta, \Box\theta, \Gamma \Rightarrow \Delta}{\Box\theta, \Gamma \Rightarrow \Delta}$ $\qquad R\Box.\quad \dfrac{\Box\Theta \Rightarrow \eta}{\Box\Theta, \Gamma \Rightarrow \Delta, \Box\eta}$

$4\Box.\quad \dfrac{\Theta, \Box\Theta \Rightarrow \eta}{\Box\Theta, \Gamma \Rightarrow \Delta, \Box\eta}$ $\qquad K\Box.\quad \dfrac{\Theta \Rightarrow \eta}{\Box\Theta, \Gamma \Rightarrow \Delta, \Box\eta}$

- G3cp: Ax, $L\bot$, $L\rightarrow$, $R\rightarrow$; G3s: G3cp with $L\Box$, $R\Box$;
- G3t: G3cp with $L\Box$, $K\Box$; G34: G3cp with $4\Box$.
- $\Box\Theta := \{\Box\theta \mid \theta \in \Theta\}$.

Prehistoric graph and prehistoric cycle

- Given a proof tree $\mathcal{T} = (T, R)$, the prehistoric graph of $\mathcal{T}$ is $\mathcal{P}(\mathcal{T}) := (F, \prec)$, where
  - F is the set of families of positive $\Box$'s in the proof tree $\mathcal{T}$,
  - (take G3s for instance) $\prec := \{\langle i, j\rangle \mid \langle(\Box\Theta(\Box_i) \Rightarrow \eta), (\Box\Theta(\Box_i), \Gamma \Rightarrow \Delta, \Box_j\eta)\rangle \in R\}$,
    - i.e., $\dfrac{\Box\Theta(\Box_i) \Rightarrow \eta}{\Box\Theta(\Box_i), \Gamma \Rightarrow \Delta, \Box_j\eta}(R\Box)$ is a step in $\mathcal{T}$.
- Given $\mathcal{T}$, a prehistoric cycle is a cycle in $\mathcal{P}(\mathcal{T})$.
- A proof $\mathcal{T}$ is cycle-free, if $\mathcal{P}(\mathcal{T})$ has no cycle.

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  - F is the set of families of positive $\Box$'s in the proof tree $\mathcal{T}$,
  - (take G3s for instance) $\prec := \{\langle i, j\rangle \mid \langle(\Box\Theta(\Box_i) \Rightarrow \eta), (\Box\Theta(\Box_i), \Gamma \Rightarrow \Delta, \Box_j\eta)\rangle \in R\}$,
    - i.e., $\dfrac{\Box\Theta(\Box_i) \Rightarrow \eta}{\Box\Theta(\Box_i), \Gamma \Rightarrow \Delta, \Box_j\eta}(R\Box)$ is a step in $\mathcal{T}$.
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Prehistoric-cycle-free fragments

- Definition:
  - Let $\mathrm{X} \in \{\mathrm{T}, \mathrm{K4}, \mathrm{S4}\}$, and $\mathrm{Y} \in \{\mathrm{G3t}, \mathrm{G34}, \mathrm{G3s}\}$, resp.;
    - $\mathrm{X}^{CF} := \{\phi \mid (\Rightarrow \phi)$ has a cycle-free proof in $\mathrm{Y}\}$.
  - For a basic embedding $(\cdot)^{\times}$:
    - $\mathrm{IPC}^{CF(\times)} := \{\mathrm{IPC} \vdash \phi \mid \phi^{\times} \in \mathrm{S4}^{CF}\}$;
    - $\mathrm{IPC}^{CF} := \bigcup_{\times} \mathrm{IPC}^{CF(\times)}$;
    - $\mathrm{IPC}_{\rightarrow}^{CF(\times)}$ and $\mathrm{IPC}_{\rightarrow}^{CF}$ are similarly defined.
- $\in CF$ is sufficient to $\in NR$ (Yu 2010 & 2014):
  - If $\mathrm{X} \in \{\mathrm{T}, \mathrm{K4}, \mathrm{S4}, \mathrm{IPC}, \mathrm{IPC}_{\rightarrow}\}$, then $\mathrm{X}^{CF} \subseteq \mathrm{X}^{NR}$.

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Properties of *CF* fragments

- Let $\mathsf{X} \in \{\mathsf{T}, \mathsf{K4}, \mathsf{S4}\}$:
  - $\phi \in \mathsf{X}^{CF}$ iff $\Box\phi \in \mathsf{X}^{CF}$ (necessitation).
  - $\phi \in \mathsf{X}^{CF}$ implies $\phi[p/\psi] \in \mathsf{X}^{CF}$ (uniform substitution).
  - X^{CF} contains:
    - $\bot \to p$.
    - $p \to (q \to p)$.
    - $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$.
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  - X^{CF} contains all axiom instances in X.
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- Let $X \in \{T, K4, S4\}$:
  - X^{NR} contains all axiom instances in X.
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- Easy to show are:
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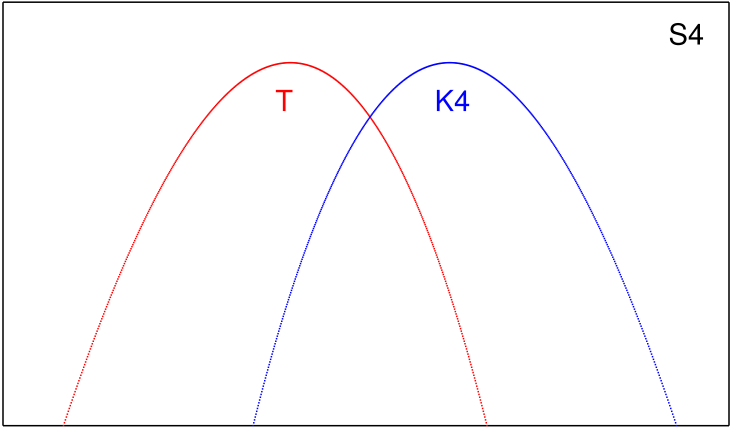
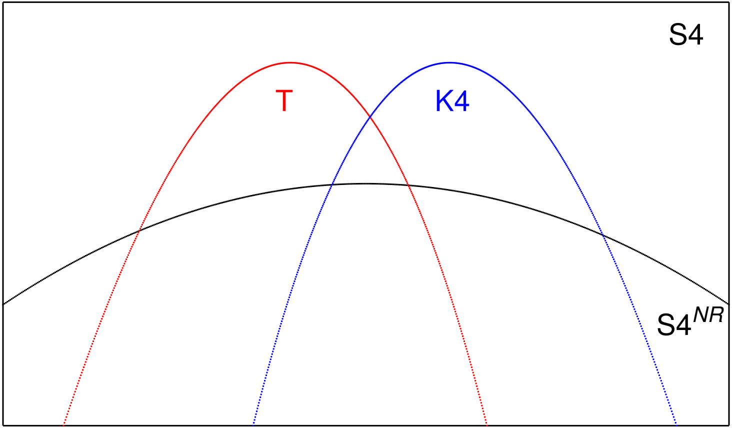


Between *NR* fragments of ML's (contiinued)



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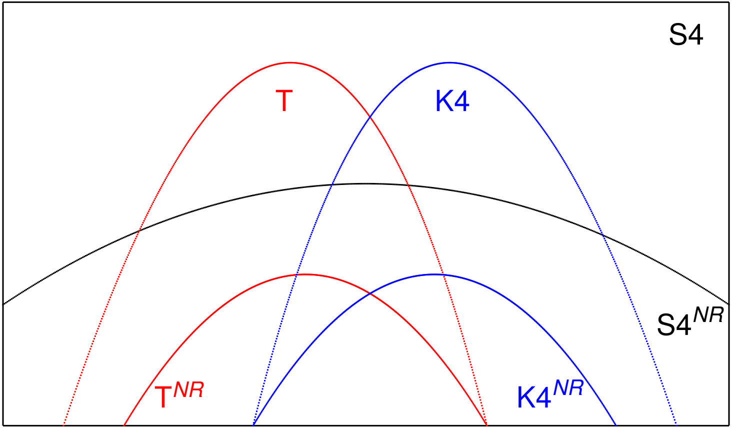

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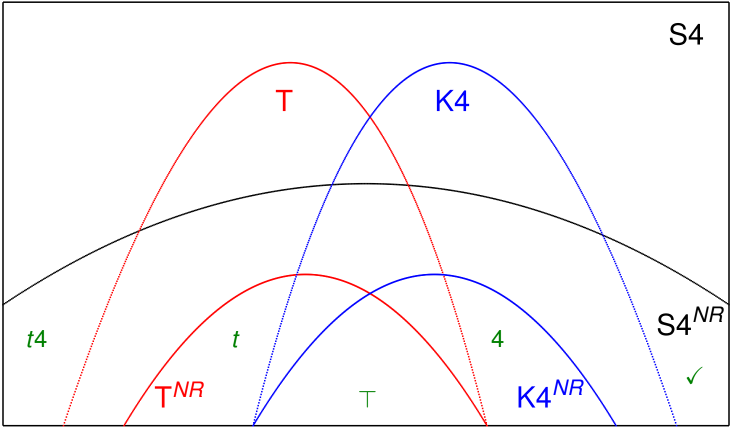

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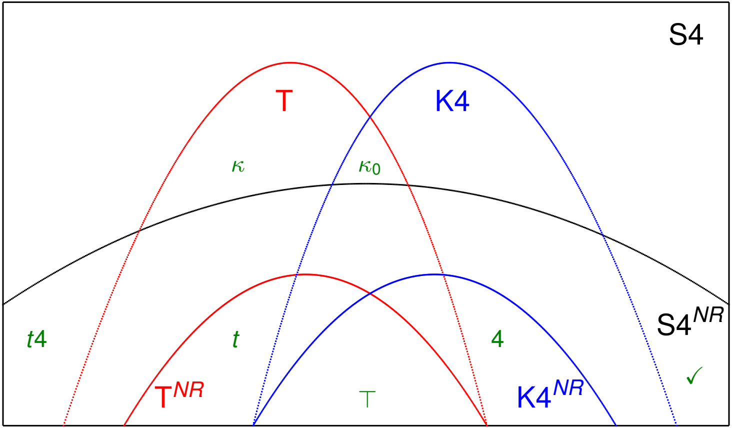

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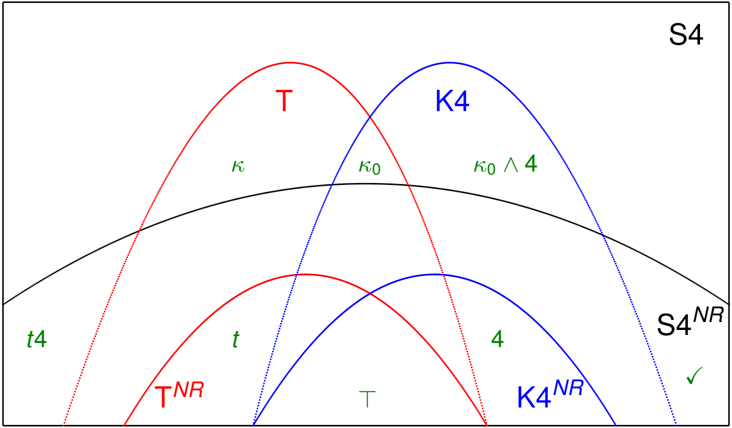

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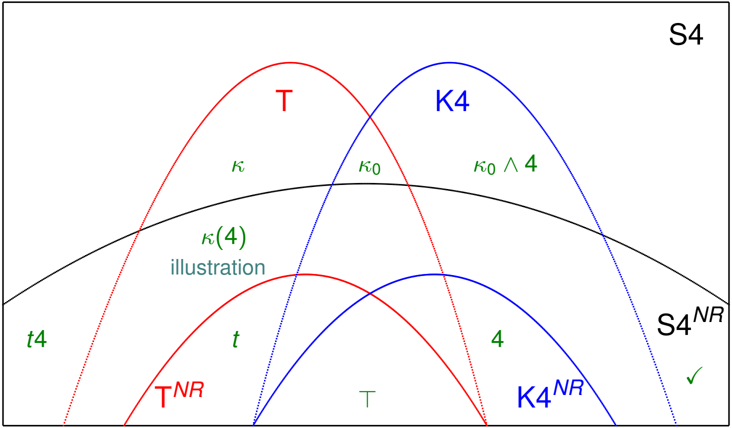

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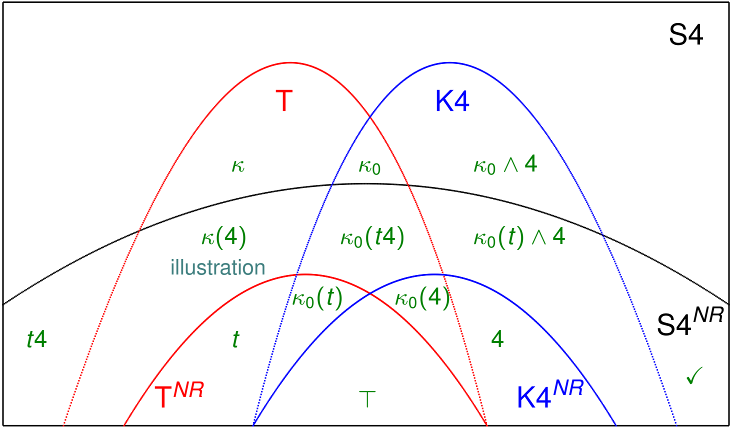

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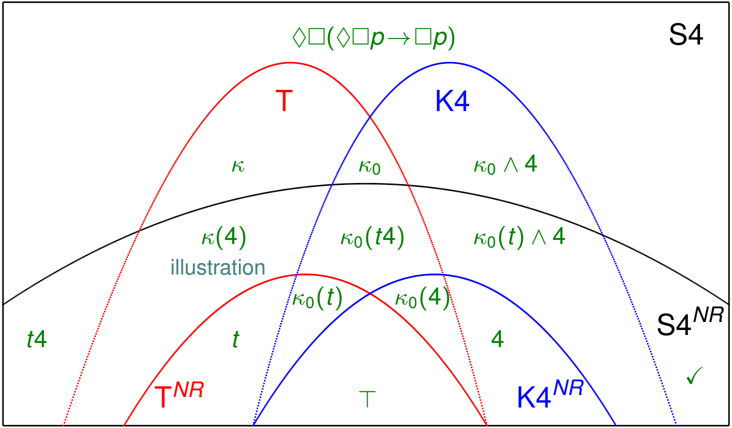

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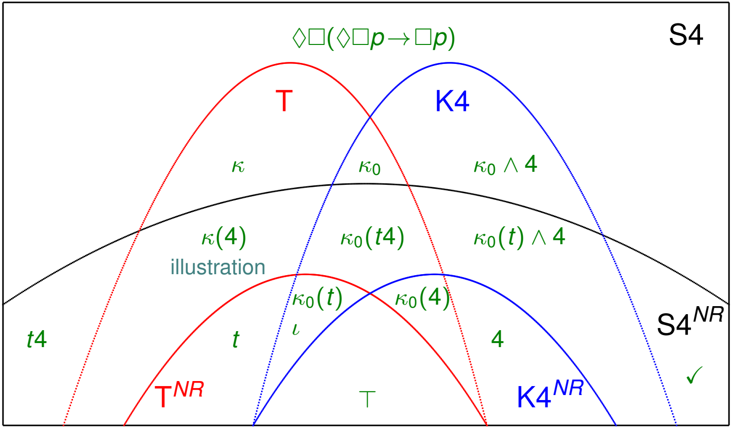

P.S.: Not all instances come from Kuznets' κ's, e.g., let $\iota = \Diamond\Box p \to \Diamond\Box\Diamond p$.

- Thanks!