Self-referentiality in the framework of justification logics

俞珺华 Yu, Junhua

Department of Philosophy, Tsinghua University

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- Realization in Justification Logic
- Self-referentiality
- Properties of non-self-referential fragments

Justification Logics JI Realization

• Realization in Justification Logic

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Justification Logics JL Realization

Justification logics JL

• Explicit versions of modal logics ML.

- $\Box \phi$ v.s. $t : \phi$,
- *t* explains contents implicitly indicated by \Box .
- Language: propositional, extended by *t*: *φ*.
 - *t* is a term (inductively defined, sensitive to logics),
 - ϕ is a formula in **this** language (where terms may occur in).
- The family of JL: >30 members, serving as explicit versions to many well-known ML's.
 - We will focus on the following five pairs:

ML K D T K4 S4

JL J JD JT J4 LP

Justification Logics JL Realization

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Justification Logics JL Realization

The Logic of Proofs LP as an example

- By Artemov in 1995.
- $\phi := \perp |\rho| \phi \rightarrow \phi |t; \phi,$ $t := c |x| t \cdot t |t+t|!t.$
- Axiom schemes:
 - Classical propositional axioms,
 - $t: \phi \rightarrow \phi$,
 - $t_1: (\phi \rightarrow \psi) \rightarrow (t_2: \phi \rightarrow t_1 \cdot t_2: \psi),$
 - $t: \phi \rightarrow !t: t: \phi$,
 - $t_1: \phi \rightarrow t_1 + t_2: \phi$ and $t_2: \phi \rightarrow t_1 + t_2: \phi$.
- Rules schemes:

- $\vdash c : A$, where c is a constant, and A is an axiom.
- Explicit version of modal logic S4.
- Formally, the implicit/explicit correspondence is called realization.

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Justification Logics JL Realization

Realization

- Realizer
 - A mapping: the language of ML ~> that of a JL;
 - Assigns a term to each \Box -occurrence in the input formula.
- Realization
 - Given realizer (·)^r and modal formula φ, the image φ^r is a potential realization;
 - ϕ^r is a realization if further $JL \vdash \phi^r$.

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Justification Logics JL Realization

Realization (continued)

- Realization theorem (Artemov 1995 & Brezhnev 2000)
 - For any modal formula ϕ :
 - Let $X \in \{K, D, T, K4, S4\}$,
 - and $Y \in \{J, JD, JT, J4, LP\},$ resp.,
 - Then what follows are equivalent:
 - $X \vdash \phi$;
 - $\mathbf{Y} \vdash \phi^r$ for some realizer $(\cdot)^r$.

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Realization in JL	In Justification Logics
Self-referentiality	In Modal Logics
Properties of NR Fragments	In Intuitionistic Propositional Logic

Self-referentiality

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Self-referential JL-formulas

• (recalled) Justification language (LP as an example)

- Formula $\phi := \bot | p | \phi \rightarrow \phi | t : \phi;$
- Term $t := c | x | t \cdot t | t + t | !t$.
- self-referential formulas like $t: \phi(t)$

• even c: A(c) is possible.

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In Justification Logics In Modal Logics In Intuitionistic Propositional Logic

Constant specification \mathcal{CS}

- Definition (take LP as our example):
 - A set of formulas of the form *c*: *A*.
- Link axioms with constants that present them in terms.
- JL(*CS*) is the fragment of JL where rule scheme *AN* can only put formulas from *CS*.
 - e.g., $JL(\emptyset)$ is the fragment of JL without AN.

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Self-referentiality of \mathcal{CS}

- Take LP as our example.
- CS is (directly) self-referential, if for some c and A

 $c: A(c) \in CS.$

- Let $CS^* := \{c: A \mid c \text{ does not occur in } A\};$
 - The largest non-self-referential constant specification.
 - Thus, JL(*CS*^{*}) is the fragment of JL where *AN* can only introduce non-self-referential formulas.

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ML^{NR}: non-self-referential realizable fragment of ML

• Definition:

- Let $X \in \{K, D, T, K4, S4\}$, and $Y \in \{J, JD, JT, J4, LP\}$, resp.;
- $X^{NR} := \{X \vdash \phi \mid Y(\mathcal{CS}^*) \vdash \phi^r \text{ for some realizer } (\cdot)^r\}.$
- A model theorem is non-self-referential if being in ML^{NR}, and self-referential otherwise.
- Self-referential modal-theorems exist. (Kuznets 2006 & 2008):
 - $K^{NR} = K$
 - $\mathsf{D}^{NR} = \mathsf{D}$
 - $\Diamond(p \to \Box p) \in \mathsf{T} \setminus \mathsf{T}^{NR}$
 - $\Box \neg (p \rightarrow \Box p) \rightarrow \Box \bot \in \mathsf{K4} \setminus \mathsf{K4}^{\mathsf{NI}}$
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Realizing intuitionistic propositional logic IPC via S4

- The initial motivation of Artemov's LP;
- The Gödel–Artemov formalization of BHK semantics;
- Gödel's modal embedding (·)[△] is a mapping from propositional language to propositional modal language that satisfies:

$$\begin{cases} p^{\triangle} = \Box p \\ \bot^{\triangle} = \Box \bot \\ (\phi \oplus \psi)^{\triangle} = \Box (\phi^{\triangle} \oplus \psi^{\triangle}) \text{ for } \oplus \in \{\land, \lor, \rightarrow\}. \end{cases}$$

 Sound-and-faithfully embeds IPC into S4, i.e., IPC ⊢ φ iff S4 ⊢ φ[△] (McKinsey & Tarski 1948).

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In Justification Logics In Modal Logics In Intuitionistic Propositional Logic

Basic embeddings

• An extension of Gödel's modal embedding.

• A potential embedding $((\cdot)^{\times})$ is basic if (let $\odot \in \{\land,\lor\}$):

$$\begin{pmatrix} \phi^{\times} = \phi^{\times}_{+} \\ p^{\times}_{+} = \Box^{h_{+}}p \quad p^{\times}_{-} = \Box^{h_{-}}p \quad \text{similar for } \bot \\ (\phi \odot \psi)^{\times}_{+} = \Box^{j_{\odot +}}(\Box^{k_{\odot +}}\phi^{\times}_{+} \odot \Box^{l_{\odot +}}\psi^{\times}_{+}) \\ (\phi \odot \psi)^{\times}_{-} = \Box^{j_{\odot -}}(\Box^{k_{\odot -}}\phi^{\times}_{-} \odot \Box^{l_{\odot -}}\psi^{\times}_{-}) \\ (\phi \rightarrow \psi)^{\times}_{+} = \Box^{j_{\rightarrow +}}(\Box^{k_{\rightarrow +}}\phi^{\times}_{+} \rightarrow \Box^{L_{\rightarrow +}}\psi^{\times}_{+}) \\ (\phi \rightarrow \psi)^{\times}_{-} = \Box^{j_{\rightarrow -}}(\Box^{k_{\rightarrow -}}\phi^{\times}_{+} \rightarrow \Box^{L_{\rightarrow -}}\psi^{\times}_{-})$$

- A basic embedding is a potential one that satisfies: IPC ⊢ φ iff S4 ⊢ φ[×].
 - possible applications on other logic pairs.

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IPC^{NR}: non-self-referential realizable fragment of IPC

• Definition:

- $\mathsf{IPC}^{\mathsf{NR}(\times)} := \{\mathsf{IPC} \vdash \phi \,|\, \phi^{\times} \in \mathsf{S4}^{\mathsf{NR}}\};$
- $\operatorname{IPC}^{NR} := \bigcup_{\times} \operatorname{IPC}^{NR(\times)};$
- An intuitionistic theorem is non-self-referential if being in IPC^{NR}, and self-referential otherwise;
- $\mathsf{IPC}_{\rightarrow}^{\mathit{NR}(\times)}$ and $\mathsf{IPC}_{\rightarrow}^{\mathit{NR}}$ are similarly defined based on $\mathsf{IPC}_{\rightarrow}.$
- Self-referential IPC-theorem exists. (Yu 2014):
 - $\{\neg \neg \alpha \mid \alpha \in \mathsf{CPC} \setminus \mathsf{IPC}\} \subseteq \mathsf{IPC} \setminus \mathsf{IPC}^{\mathsf{NR}}$
 - $((((p \rightarrow q) \rightarrow p) \rightarrow p) \rightarrow q) \rightarrow q \in \mathsf{IPC}_{\rightarrow} \setminus \mathsf{IPC}_{\rightarrow}^{NR}$

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Realization in JL	A Wieldy Tool
Self-referentiality	Failures of MP
Properties of NR Fragments	Between NR fragments of ML's

Properties of non-self-referential realizable fragments

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A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Prehistoric-cycle-free provable fragment

- For each logic mentioned above,
 - the *CF* (prehistoric-cycle-free provable) fragment is a subset of

the NR (non-self-referential realizable) fragment;

- The best known approximation;
- Decidable, wieldy for simple formulas.

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A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

The underline calculus G3[st4]

Ax.
$$\overline{\rho,\Gamma\Rightarrow\Delta,\rho}$$
 $L\perp$. $\overline{\perp,\Gamma\Rightarrow\Delta}$ $L\rightarrow$. $\Gamma\Rightarrow\Delta,\phi~\psi,\Gamma\Rightarrow\Delta$ $R\rightarrow$. $\overline{\psi,\Gamma\Rightarrow\Delta,\psi}$ $L\rightarrow$. $\overline{\psi,\Gamma\Rightarrow\Delta,\phi\to\psi}$ $R\rightarrow$. $\overline{\psi,\Gamma\Rightarrow\Delta,\psi\to\psi}$ $L\square$. $\frac{\theta,\Box\theta,\Gamma\Rightarrow\Delta}{\Box\theta,\Gamma\Rightarrow\Delta}$ $R\square$. $\frac{\Box\Theta\Rightarrow\eta}{\Box\Theta,\Gamma\Rightarrow\Delta,\Box\eta}$ $4\Box$. $\frac{\Theta,\Box\Theta\Rightarrow\eta}{\Box\Theta,\Gamma\Rightarrow\Delta,\Box\eta}$ $K\Box$. $\frac{\Theta\Rightarrow\eta}{\Box\Theta,\Gamma\Rightarrow\Delta,\Box\eta}$

- G3cp: $Ax, L \perp, L \rightarrow, R \rightarrow$;
- G3t: G3cp with $L\Box$, $K\Box$;
- $\Box \Theta := \{ \Box \theta \mid \theta \in \Theta \}.$

G3s: G3cp with $L\Box$, $R\Box$; G34: G3cp with $4\Box$.

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A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Prehistoric graph and prehistoric cycle

- Given a proof tree $\mathcal{T} = (T, R)$, the prehistoric graph of \mathcal{T} is $\mathcal{P}(\mathcal{T}) := (F, \prec)$, where
 - *F* is the set of families of positive \Box 's in the proof tree T,

• (take G3s for instance)

$$\prec := \{ \langle i, j \rangle \mid \langle (\Box \Theta(\Box_i) \Rightarrow \eta), (\Box \Theta(\Box_i), \Gamma \Rightarrow \Delta, \Box_j \eta) \rangle \in R \},$$
• i.e., $\frac{\Box \Theta(\Box_i) \Rightarrow \eta}{\Box \Theta(\Box_i), \Gamma \Rightarrow \Delta, \Box_j \eta} (R \Box)$ is a step in \mathcal{T} .

- Given \mathcal{T} , a prehistoric cycle is a cycle in $\mathcal{P}(\mathcal{T})$.
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A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Prehistoric-cycle-free fragments

Definition:

- Let $X \in \{T, K4, S4\}$, and $Y \in \{G3t, G34, G3s\}$, resp.;
 - $X^{CF} := \{\phi \mid (\Rightarrow \phi) \text{ has a cycle-free proof in } Y\}.$
- For a basic embedding $(\cdot)^{\times}$:
 - $\operatorname{IPC}_{\operatorname{CF}}^{\operatorname{CF}(\times)} := \{\operatorname{IPC} \vdash \phi \mid \phi^{\times} \in \operatorname{S4}^{\operatorname{CF}}\};$
 - $\mathsf{IPC}^{CF} := \bigcup_{\times} \mathsf{IPC}^{CF(\times)};$
 - $\bullet~ \mathsf{IPC}_{\rightarrow}^{\mathit{CF}(\times)}$ and $\mathsf{IPC}_{\rightarrow}^{\mathit{CF}}$ are similarly defined.
- $\in CF$ is sufficient to $\in NR$ (Yu 2010 & 2014):
 - If $X \in \{T, K4, S4, IPC, IPC_{\rightarrow}\}$, then $X^{CF} \subseteq X^{NR}$.

A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

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A Wieldy Tool Failures of MP Between NR fragments of ML's

Properties of CF fragments

• Let $X \in \{T, K4, S4\}$:

- $\phi \in X^{CF}$ iff $\Box \phi \in X^{CF}$ (necessitation).
- $\phi \in X^{CF}$ implies $\phi[p/\psi] \in X^{CF}$ (uniform substitution).
- X^{CF} contains:

$$-\perp \rightarrow p$$

$$-p
ightarrow (q
ightarrow p).$$

$$-(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$$

$$-((p \rightarrow q) \rightarrow p) \rightarrow p.$$

$$-\Box(\rho \rightarrow q) \rightarrow (\Box p \rightarrow \Box q).$$

$$-\Box p \rightarrow p$$
 (for T, S4).

$$-\Box p \rightarrow \Box \Box p$$
 (for K4, S4).

- X^{CF} contains all axiom instances in X.
 - Applying uniform substitution to the above.

• $\alpha \rightarrow (\beta \rightarrow \alpha), (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)) \in \mathsf{IPC}_{\rightarrow}^{CF}$.

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$$- \Box (p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q).$$

$$- \Box p \rightarrow p \qquad (for T S4)$$

- $-\Box p \rightarrow p$ (for T, S4).
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- X^{CF} contains all axiom instances in X.
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A Wieldy Tool Failures of MP Between NR fragments of ML's

Applied to NR fragments

• Let $X \in \{T, K4, S4\}$:

• X^{NR} contains all axiom instances in X.

– by the fact that $X^{CF} \subseteq X^{NR}$.

- X^{NR} is closed under necessitation.
 - directly by Artemov's proof of internalization theorem.
- X^{NR} is not closed under MP.

- otherwise $X^{NR} = X$, contradiction.

• Thus, non-self-referentiality can be abnormal. (Yu 2017)

• IPC^{NR} is not closed under MP.

- $\alpha \rightarrow (\beta \rightarrow \alpha), (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)) \in \mathsf{IPC}_{\rightarrow}^{NR}.$ - by the fact that $\mathsf{IPC}_{\rightarrow}^{CF} \subseteq \mathsf{IPC}_{\rightarrow}^{NR}.$
- IPC^{*NR*} is not closed under *MP*.
 - otherwise IPC $_{\rightarrow}^{NR}$ = IPC $_{\rightarrow}$, contradiction.
- So is IPC^{NR}

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A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Between NR fragments of ML's

• Easy to show are:

- $T^{NR} \subseteq S4^{NR}$ and
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 - though we will not give a proof here...
- Hard to show is:
 - there are no more inclusions!
- Therefore, when going from a smaller ML to a greater ML, non-self-referentiality is not always conservative. (Yu 2017)

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A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Between NR fragments of ML's (continued)



俞珺华 (Yu, Junhua) Self-referentiality in the framework of justification logics

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A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Between NR fragments of ML's (contiinued)



俞珺华 (Yu, Junhua) Self-referentiality in the framework of justification logics

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A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Between NR fragments of ML's (contiiinued)



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Realization in JL A Wi Self-referentiality Failu Properties of *NR* Fragments Betw

A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Between NR fragments of ML's (contivnued)



A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Between NR fragments of ML's (contvnued)



Realization in JL A Wieldy Self-referentiality Failures Properties of *NR* Fragments Between

A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Between NR fragments of ML's (contvinued)



Realization in JL A Wieldy Tool Self-referentiality Failures of *MP* Properties of *NR* Fragments Between *NR* fragments of ML's

Between NR fragments of ML's (contviinued)



Realization in JL A Wieldy Tool Self-referentiality Failures of MP Properties of NR Fragments Between NR fragments of ML's

Between NR fragments of ML's (contviiinued)



Realization in JL A Wield Self-referentiality Failures Properties of NR Fragments Betwee

A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Between NR fragments of ML's (contixnued)



Realization in JL A Wie Self-referentiality Failure Properties of *NR* Fragments Betwee

A Wieldy Tool Failures of *MP* Between *NR* fragments of ML's

Between NR fragments of ML's (contxnued)



P.S.: Not all instances come from Kuznets' κ 's, e.g., let $\iota = \Diamond \Box p \rightarrow \Diamond \Box \Diamond p$.

Realization in JL	A Wieldy Tool
Self-referentiality	Failures of MP
Properties of NR Fragments	Between NR fragments of ML's

Thanks!

俞珺华 (Yu, Junhua) Self-referentiality in the framework of justification logics

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